Isomorphism Theorems

To try to understand objects one can study quotients. The isomorphism theorems are "basic" tools for dealing with quotients. I once attended a conference where the speaker referred to the First Isomorphism Theorem (also called the Fundamental Theorem of Homomorphisms) as a theorem we "learn in our childhood." While this and following theorems might look intimidating at first, once we are used to them, they are easy to apply and very useful.

While we will discuss these theorems in the context of group theory, it is worth mentioning that they hold in every branch of algebra. If we wanted to state and prove these results in the context of rings or vector spaces or Lie algebras or whatever, the statements would look nearly identical and the proofs would be structured the same way (but with different notations).

Before stating and proving the First Isomorphism Theorem for groups, let us look at the set theory version.

Theorem: Let $f : A \to B$ be a function.

Then f can be factored as $f = \iota \circ \overline{f} \circ \pi$ where π is onto, $\overline{f}$ is invertible, and ι is one-to-one.

Proof: First, we will illustrate with an example. Suppose $A = \{1, 2, 3, 4, 5\}$ and $B = \{a, b, c, d\}$ and let $f(1) = f(2) = a$, $f(3) = f(4) = b$, and $f(5) = c$. We let $\overline{A}$ be the *fibers* of f. In other words, each element of $\overline{A}$ is a set of all values that map to a particular output. In particular, $\overline{A} = \{\{1, 2\}, \{3, 4\}, \{5\}\}$ (these are the sets of elements that map to a, b, and c respectively). Then we define $\pi : A \to \overline{A}$ to be the projection onto the fibers: $\pi(1) = \pi(2) = \{1, 2\}$, $\pi(3) = \pi(4) = \{3, 4\}$, and $\pi(5) = \{5\}$. Notice that π maps A onto $\overline{A}$. Next, define $\overline{f} : \overline{A} \to \text{range}(f)$ by $\overline{f}(\{1, 2\}) = a$, $\overline{f}(\{3, 4\}) = b$, and $\overline{f}(\{5\}) = c$. Notice that $\overline{f}$ is an invertible map between $\overline{A} = \{\{1, 2\}, \{3, 4\}, \{5\}\}$ and range $= f(A) = \{a, b, c\}$. Finally, let $\iota : \text{range}(f) \to B$ be the *inclusion* map defined $\iota(a) = a$, $\iota(b) = b$, and $\iota(c) = c$ is one-to-one. Notice that $f = \iota \circ \overline{f} \circ \pi$ (for example, $\iota(\overline{f}(\pi(1))) = \iota(\overline{f}(\{1, 2\})) = \iota(a) = a = f(1)$).

Now the general proof. Let $f : A \to B$ be a function. Consider the relation on A defined by $x \sim y$ if and only if $f(x) = f(y)$. This is an equivalence relation: $f(x) = f(x)$ so $x \sim x$; $x \sim y$ implies $f(x) = f(y)$ so $f(y) = f(x)$ so $y \sim x$; $x \sim y$ and $y \sim z$ implies $f(x) = f(y)$ and $f(y) = f(z)$ so $f(x) = f(z)$ so $x \sim z$. Thus the equivalence classes of this relation partition A. Let $[x]$ denote the equivalence class of x. This is $\{y \in A \mid x \sim y\} = \{y \in A \mid f(x) = f(y)\} = \{y \in A \mid f(y) \in \{f(x)\}\} = f^{-1}(\{f(x)\})$ (i.e., the inverse image of the singleton set $f(x)$ – in other words, the fiber over $f(x)$). Let $\overline{A} = \{[x] \mid x \in A\}$ (i.e., the set of equivalence classes – that is – the fibers of f).

Define $\pi : A \to \overline{A}$ by $\pi(x) = [x]$ (map each element to its equivalence class). Clearly, π is onto. Next, let $\overline{f} : \overline{A} \to f(A)$ be defined by $\overline{f}([x]) = f(x)$ (i.e., map each fiber to the element it is the fiber over). Notice that $[x] = [y]$ iff $x \sim y$ iff $f(x) = f(y)$ iff $\overline{f}([x]) = \overline{f}([y])$. Reading in one direction, this says equal inputs yield equal outputs (i.e., $\overline{f}$ is well-defined). Reading in the other direction, this says equal outputs imply equal inputs (i.e., $\overline{f}$ is one-to-one). Notice that $y \in f(A)$ implies there is some $x \in A$ such that $f(x) = y$. Thus $\overline{f}([x]) = f(x) = y$ and so $\overline{f}$ is also onto. We now have $\overline{f}$ is an invertible function. Now let $\iota : f(A) \to B$ be defined by $\iota(x) = x$ (this kind of map is called an *inclusion* map). Obviously ι is one-to-one. Finally, notice that for any $x \in A$, $(\iota \circ \overline{f} \circ \pi)(x) = \iota(\overline{f}(\pi(x))) = \iota(\overline{f}([x])) = \iota(f(x)) = f(x)$ so that $\iota \circ \overline{f} \circ \pi = f$. ■

The First Isomorphism Theorem is essentially just the above theorem applied to group homomorphisms. Let us see what the fibers of a group homomorphism are. Let $\varphi : G \to H$ be a homomorphism of groups. Recall that the kernel of φ is the set of elements that map to the identity of the codomain (i.e., H). In other words, $\text{Ker}(\varphi) = \varphi^{-1}(\{1\})$. So the kernel is the fiber over $\varphi(1) = 1$. Let us identify the other fibers of φ:

$$\begin{aligned}\varphi^{-1}(\{\varphi(a)\}) = \{x \in G \mid \varphi(x) = \varphi(a)\} &= \{x \in G \mid 1 = \varphi(x)^{-1}\varphi(a) = \varphi(x^{-1}a)\} = \{x \in G \mid x^{-1}a \in \text{Ker}(\varphi)\} \\ &= \{x \in G \mid x\text{Ker}(\varphi) = a\text{Ker}(\varphi)\} = \{x \in G \mid x \in a\text{Ker}(\varphi)\} = a\text{Ker}(\varphi)\end{aligned}$$

In other words, a and b both map to the same output if and only if they "differ" by a kernel element:

$\varphi(a) = \varphi(b)$ if and only if there exists $k \in \text{Ker}(\varphi)$ such that $b = ak$.

The set of fibers is our set of (left) cosets $G/\text{Ker}(\varphi)$. We now define our map between fibers and the range:

$$\overline{\varphi} : G/\text{Ker}(\varphi) \to \varphi(G) \quad \text{defined by} \quad \overline{\varphi}(a\text{Ker}(\varphi)) = \varphi(a)$$

Notice that $a\text{Ker}(\varphi) = b\text{Ker}(\varphi)$ if and only if $\varphi(a) = \varphi(b)$. Thus $\overline{\varphi}$ is a well-defined, one-to-one function. In addition, we have rigged it up to be onto. Also, since $\text{Ker}(\varphi)$ is a normal subgroup of G, $G/\text{Ker}(\varphi)$ is itself a (quotient) group. Notice that $\overline{\varphi}(a\text{Ker}(\varphi)b\text{Ker}(\varphi)) = \overline{\varphi}(ab\text{Ker}(\varphi)) = \varphi(ab) = \varphi(a)\varphi(b) = \overline{\varphi}(a\text{Ker}(\varphi))\overline{\varphi}(b\text{Ker}(\varphi))$ so that $\overline{\varphi}$ is operation preserving (i.e., a homomorphism). Thus $\overline{\varphi}$ is an isomorphism. Once can easily check that $\pi : G \to G/\text{Ker}(\varphi)$ defined by $\pi(g) = g\text{Ker}(\varphi)$ is an onto homomorphism (i.e., an epimorphism) and $\iota : \varphi(G) \to H$ defined by $\iota(h) = h$ is a one-to-one homomorphism (i.e., a monomorphism). Thus $\varphi = \iota \circ \overline{\varphi} \circ \pi$ can be factored into a epi- followed by iso- followed by mono-morphism. In particular, we have shown:

Theorem: [First Isomorphism Theorem] Let $\varphi : G \to H$ be a homomorphism. Then $G \big/ \mathrm{Ker}(\varphi) \cong \varphi(G)$ (i.e., the domain mod the kernel is isomorphic to the image).

One of many important consequences of the First Isomorphism Theorem is that we can identify the concepts of *quotient* and *homomorphic image* as well as *normal subgroup* and *kernel*. First, we note that given a homomorphism, $\varphi : G \to H$, its range (=image), $\varphi(G)$ is a subgroup of the codomain, H. We call $\varphi(G)$ a homomorphic image of G.

To tie a few things together, recall that we defined a projection map π while factoring our homomorphism φ. We generalize. Suppose G is a group with normal subgroup N. Define $\pi : G \to G/N$ where $\pi(g) = gN$ to be a projection of G onto the quotient group G/N. This homomorphism ($\pi(ab) = abN = aNbN = \pi(a)\pi(b)$) is obviously onto and $\mathrm{Ker}(\pi) = \{g \in G \mid \pi(g) = N\} = \{g \in G \mid gN = N\} = \{g \in G \mid g \in N\} = N$. Therefore, since the image of π is G/N, we have that G/N is a homomorphic image. Conversely the First Isomorphism Theorem says that homomorphic images are isomorphic to the quotient of the domain by the kernel. We summarize:

Corollary: Every homomorphic image is (up to isomorphism) a quotient. Conversely, every quotient is a homomorphic image. Also, the kernel of a homomorphism is a normal subgroup (of the homomorphism's domain). Conversely, every normal subgroup is the kernel of some homomorphism.

As the name indicates, the First Isomorphism Theorem is the first in a family of theorems. The next theorem, the Second Isomorphism Theorem, is also called the...

Theorem: [Diamond Isomorphism Theorem] Let G have subgroups A and B where B is a normal subgroup of G. Then (1) $AB = \{ab \mid a \in A \text{ and } b \in B\}$ is a subgroup of G with B as a normal subgroup, (2) $A \cap B$ is a normal subgroup of A, and (3) $AB/B \cong A/(A \cap B)$.

Note: In the diagram to the right, the quotients of groups by their subgroups connected by a double line are isomorphic to each other. Thus giving rise to this theorem's name.

$$\begin{tikzcd}[column sep=small] & AB \arrow[dl, dash] \arrow[dr, equal] & \\ A \arrow[dr, equal] & & B \arrow[dl, dash] \\ & A \cap B & \end{tikzcd}$$

Proof: For (1), let $ab, cd \in AB$ where $a, c \in A$ and $b, d \in B$. Then, noting $c^{-1}bc \in B$ because B is normal, we have $ab \cdot cd = ac \cdot c^{-1}bcd \in AB$ considering $a, c \in A$ and $c^{-1}bc, d \in B$ and the fact the A and B are subgroups and thus closed under the operation. Also, $(ab)^{-1} = b^{-1}a^{-1} = a^{-1} \cdot ab^{-1}a^{-1} \in AB$ again because $a^{-1} \in A$ (subgroups are closed under inversion) and $ab^{-1}a^{-1} \in B$ (normal subgroups are closed under inversion and conjugation). Obviously, $B = 1 \cdot B$ is a subset of AB and since B is normal in G, it is also normal in the subgroup AB.

For (2), we know that $A \cap B$ is a subgroup of A, we just need to check normality. Let $x \in A \cap B$ and $g \in A$. Then $x \in A$ and $x \in B$. We have $gxg^{-1} \in A$ (by closure under the operation and inverses since $x, g \in A$) and $gxg^{-1} \in B$ since B is a normal subgroup of G. Therefore, $gxg^{-1} \in A \cap B$ and thus it is a normal subgroup of A.

Now we prove (3) by utilizing the First Isomorphism Theorem. Consider $\varphi : A \to AB/B$ defined by $\varphi(x) = xB$. Notice if $x \in A$, then $x = x \cdot 1 \in AB$ thus $xB \in AB/B$ [Our codomain is sensible]. Also, $\varphi(xg) = xgB = xBgB = \varphi(x)\varphi(g)$ so φ is a homomorphism. Also, consider $abB \in AB/B$ where $a \in A$ and $b \in B$. Then $\varphi(a) = aB = abB$ (since b is absorbed by B). Thus φ is onto. Finally, $\mathrm{Ker}(\varphi) = \{a \in A \mid \varphi(a) = B\} = \{a \in A \mid aB = B\} = \{a \in A \mid a \in B\} = A \cap B$. Now apply the First Isomorphism Theorem: $A/(A \cap B) = A/\mathrm{Ker}(\varphi) \cong \varphi(A) = AB/B$. ■

The next isomorphism theorem reveals that quotients of quotients are just quotients!

Theorem: [Third Isomorphism Theorem] Let $A \lhd G$. Suppose $\mathcal{B} \lhd G/A$. Then $B = \{x \in G \mid xA \in \mathcal{B}\} \lhd G$. Moreover, $A \subseteq B$, $B/A = \mathcal{B}$ (i.e., normal subgroups of G/A come from quotienting normal subgroups of G), and $(G/A)/(B/A) \cong G/B$ (like canceling fractions – but not).

Proof: Recall that A is the identity of G/A so that $A \in \mathcal{B}$ (subgroups contain the identity). Thus $a \in A$ implies $aA = A \in \mathcal{B}$ so that $a \in B$ and so $A \subseteq B$. Let $x, y \in B$ and $g \in G$. Therefore, $xA, yA \in \mathcal{B}$ and so $xyA = xAyA, x^{-1}A = (xA)^{-1}, (gxg^{-1})A = gAxA(gA)^{-1} \in \mathcal{B}$ since $\mathcal{B}$ is a normal subgroup of G/A. Therefore, $xy, x^{-1}, gxg^{-1} \in B$ and thus B is a normal subgroup of G. Note that $B/A = \{xA \mid x \in B\} = \{xA \mid xA \in \mathcal{B}\} = \mathcal{B}$.

To establish our last fact, once again we appeal to the First Isomorphism Theorem. Let $\varphi : G/A \to G/B$ be defined by $\varphi(gA) = gB$. Since we just "defined" a map in terms of a representative of a coset, we need to make sure it is well-defined. To that end, suppose $gA = xA$. Then $g^{-1}x \in A$ but $A \subseteq B$ so $g^{-1}x \in B$ and thus $gB = xB$ so φ is a well-defined function. Next, $\varphi(xAgA) = \varphi(xgA) = xgB = xBgB = \varphi(xA)\varphi(gA)$ so φ is a homomorphism. Given $gB \in G/B$, we have $gA \in G/A$ where $\varphi(gA) = gB$ so φ is onto. Also, keeping in mind that B is the identity of G/B, $\mathrm{Ker}(\varphi) = \{xA \in G/A \mid \varphi(xA) = B\} = \{xA \in G/A \mid xB = B\} = \{xA \in G/A \mid; x \in B\} = B/A$. Therefore, by the First Isomorphism Theorem, $(G/A)/(B/A) = (G/A)/\mathrm{Ker}(\varphi) \cong \varphi(G/A) = G/B$. ■

Our last isomorphism theorem, allows us to translate between information about the subgroups of a group and its quotients. Keeping in mind that quotients and homomorphic images are really essentially the same thing (up to isomorphism), we can either state this theorem in terms of quotients or homomorphisms. We will state it in terms of the latter. This Fourth Isomorphism Theorem is also known as the...

Theorem: [Lattice Isomorphism Theorem] Let $\varphi : G \to H$ be a homomorphism with kernel $K = \mathrm{Ker}(\varphi)$. There is a bijection (i.e., one-to-one and onto map) between the subgroups of G which contain the kernel K and the subgroups of the range $\varphi(G)$: $K \subseteq A \subseteq G$ maps to $\{1\} \subseteq \varphi(A) \subseteq \varphi(G)$. Moreover, for any subgroups A and B of G which contain K, we have (1) $A \subseteq B$ if and only if $\varphi(A) \subseteq \varphi(B)$, (2) if $A \subseteq B$, then $[B : A] = [\varphi(B) : \varphi(A)]$, (3) $\varphi(\langle A, B\rangle) = \langle \varphi(A), \varphi(B)\rangle$, (4) $\varphi(A \cap B) = \varphi(A) \cap \varphi(B)$, and (5) $A \triangleleft G$ if and only if $\varphi(A) \triangleleft \varphi(G)$.

Note: Recall that $\langle A, B\rangle$ is the smallest subgroup containing both A and B (i.e., the subgroup generated by $A \cup B$).

Proof: Let $\mathcal{L} = \{A \subseteq G \mid A$ is a subgroup of G and $K \subseteq A\}$ and $\mathcal{M} = \{C \subseteq H \mid C$ is a subgroup of $\varphi(G)\}$ and recall define $\Phi : \mathcal{L} \to \mathcal{M}$ by $\Phi(A) = \varphi(A) = \{\varphi(x) \mid x \in A\}$ (i.e., mapping sets to their images under φ). We know that the image of a subgroup is a subgroup of the range, so if $A \in \mathcal{L}$, then $\Phi(A) \in \mathcal{M}$ [This map makes sense]. We claim that $\Phi^{-1} : \mathcal{M} \to \mathcal{L}$ is given by $\Phi^{-1}(C) = \varphi^{-1}(C) = \{g \in G \mid \varphi(g) \in C\}$ (i.e., the inverse image of C under φ). Suppose $C \in \mathcal{M}$, we know that inverse images of subgroups are subgroups, so $\varphi^{-1}(C)$ is a subgroup. Moreover, if $k \in K = \mathrm{Ker}(\varphi)$, then $\varphi(k) = 1 \in C$ so $k \in \varphi^{-1}(C)$ (i.e., $K \subseteq \varphi^{-1}(C)$). Therefore, $\varphi^{-1}(C) \in \mathcal{L}$. Notice that $\varphi(\varphi^{-1}(C)) = \varphi(\{g \in G \mid \varphi(g) \in C\}) = \{\varphi(g) \mid \varphi(g) \in C\} = C$ since every element of C is the image of something in G (i.e., $C \subseteq \varphi(G)$). Next, suppose $A \in \mathcal{L}$. Then $\varphi^{-1}(\varphi(A)) = \{g \in G \mid \varphi(g) \in \varphi(A)\}$. This is the union of all of the fibers over elements of $\varphi(A)$. Recall that the fiber over $\varphi(a)$ is $a\mathrm{Ker}(\varphi) = aK$. Therefore, $\varphi^{-1}(\varphi(A)) = \cup_{a \in A} aK = AK = A$ since $K \subseteq A$. We have just shown that $\Phi(A) = \varphi(A)$ and $\Phi^{-1}(C) = \varphi^{-1}(C)$ are inverse maps of each other. Therefore, Φ is a bijection between $\mathcal{L}$ and $\mathcal{M}$.

For (1), clearly $A \subseteq B$ implies $\varphi(A) \subseteq \varphi(B)$, and $\varphi(A) \subseteq \varphi(B)$ implies $A = \varphi^{-1}(\varphi(A)) \subseteq \varphi^{-1}(\varphi(B)) = B$.

For (2), consider $f : B/A \to \varphi(B)/\varphi(A)$ defined by $f(xA) = \varphi(x)\varphi(A)$. Notice that $xA = yA$ (where $x, y \in B$) implies $x^{-1}y \in A$ so that $\varphi(x)^{-1}\varphi(y) = \varphi(x^{-1}y) \in \varphi(A)$ and thus $\varphi(x)\varphi(A) = \varphi(y)\varphi(A)$. Conversely, suppose $\varphi(x)\varphi(A) = \varphi(y)\varphi(A)$ (where $x, y \in B$). This implies $\varphi(x^{-1}y) = \varphi(x)^{-1}\varphi(y) \in \varphi(A)$ and thus $\varphi(x^{-1}y) = \varphi(a)$ for some $a \in A$. Therefore, $x^{-1}y = ak$ for some $k \in \mathrm{Ker}(\varphi) = K$. But $K \subseteq A$ so $x^{-1}y \in A$ and thus $xA = yA$. In other words, f is a well-defined, one-to-one function (and obviously onto). Thus the cardinalities of B/A and $\varphi(B)/\varphi(A)$ must be equal.

For (3), recall that elements of the subgroup generated by S can be expressed as words over the alphabet $S \cup S^{-1}$. Noting that subgroups are closed under inverses, elements of $\langle A, B\rangle$ are just words over $A \cup B$ and elements of $\langle \varphi(A), \varphi(B)\rangle$ are just words over $\varphi(A) \cup \varphi(B)$. Let $w = w_1 w_2 \cdot w_\ell$ where w_k's belong to $A \cup B$. Then $\varphi(w) = \varphi(w_1)\varphi(w_2)\cdots\varphi(w_\ell)$ so that the image of a word over $A \cup B$ is just a word over $\varphi(A) \cup \varphi(B)$ since $\varphi(w_k)$'s belong to $\varphi(A) \cup \varphi(B)$. The result now follows.

For (4), if $g \in \varphi(A \cap B)$ then there is some $x \in A \cap B$ such that $g = \varphi(x)$. Then since $x \in A$ and $x \in B$, we have $g = \varphi(x) \in \varphi(A) \cap \varphi(B)$. Conversely, suppose $g \in \varphi(A) \cap \varphi(B)$. Then because $g \in \varphi(A)$ (resp. $g \in \varphi(B)$) there is some $a \in A$ such that $g = \varphi(a)$ (resp. $b \in B$ such that $g = \varphi(b)$). Now $\varphi(a) = g = \varphi(b)$ implies $a = bk$ for some $k \in \mathrm{Ker}(\varphi) = K$. But $K \subseteq B$. Therefore, $a = bk \in B$ thus $a \in A \cap B$ and so $g = \varphi(a) \in \varphi(A \cap B)$.

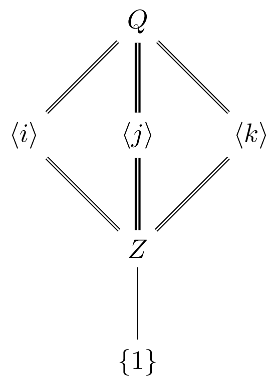
Finally, for (5), suppose $A \triangleleft G$. Let $\varphi(g) \in \varphi(G)$ and $\varphi(a) \in \varphi(A)$ (where $g \in G$ and $a \in A$). Then $\varphi(g)\varphi(a)\varphi(g)^{-1} = \varphi(gag^{-1}) \in \varphi(A)$ because $gag^{-1}\ inA$ (normal subgroups are closed under conjugation). Thus $\varphi(A) \triangleleft \varphi(G)$. Conversely, suppose $\varphi(A) \triangleleft \varphi(G)$. Let $g \in G$ and $a \in A$. Then $\varphi(gag^{-1}) = \varphi(g)\varphi(a)\varphi(g)^{-1} \in \varphi(A)$ because the normal subgroup $\varphi(A)$ is closed under conjugation. Therefore, $\varphi(gag^{-1}) = \varphi(x)$ for some $x \in A$. But this implies that $gag^{-1} = xk$ for some $k \in K$. Finally, $K \subseteq A$ so that $gag^{-1} = xk \in A$ and so A is normal in G.■

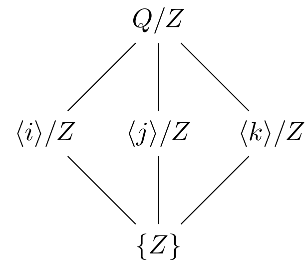
Why *lattice*? A lattice L is a partially ordered set [a partial order is a relation that is reflexive: $x \leq x$, anti-symmetric: $x \leq y$ and $y \leq x$ implies $x = y$, and transitive: $x \leq y$ and $y \leq z$ implies $x \leq z$] equipped with a notion of meet and join – like a min and max [$x \wedge y \leq x$ and $x \wedge y \leq y$ and if $z \leq x$ and $z \leq y$ we have $z \leq x \wedge y$, likewise $x \leq x \vee y$ and $y \leq x \vee y$ and when $x \leq z$ and $y \leq z$ we have $x \vee y \leq z$]. In algebra, generally each object has a corresponding lattice of subobjects. Notice that the subgroups of a group are partially ordered by set inclusion: $\subseteq$, given two subgroups, say A and B, their meet is the subgroup $A \cap B$ and their join is the subgroup generated by $A \cup B$ (i.e., $\langle A, B\rangle$). In our theorem above, $\mathcal{L}$ and $\mathcal{M}$ are lattices. Notice that both $\mathcal{L}$ and $\mathcal{M}$ even have maximum and minimum elements. The max of $\mathcal{L}$ is G and the min is K. The max of $\mathcal{M}$ is $\varphi(G)$ and the min is $\{1\}$.

If one develops a theory of lattices, one can speak of lattice morphisms and isomorphisms. A lattice morphism is a function between lattices that preserves the order relation and sends meets to meets and joins to joins. If such a map is invertble, then we call it a lattice isomorphism. Notice that in the theorem above (1), (3), and (4) tell us that Φ is a lattice isomorphism. Thus most of the Lattice Isomorphism Theorem is just telling us that the lattice of subgroups of G containing the kernel is isomorphic to the lattice of subgroups of the range.

We conclude with a few examples applying the Lattice Isomorphism Theorem to quotients (i.e., applying it to projection homomorphism $\pi : G \to G/N$ for some normal subgroup N).

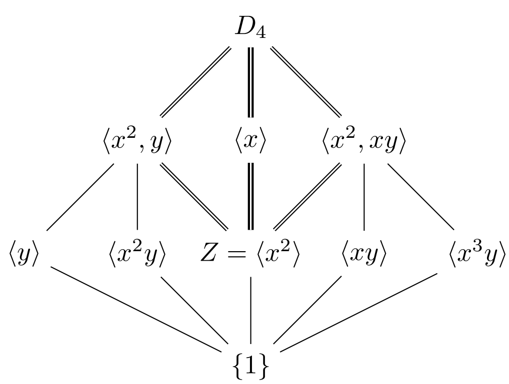
Example: Let $Q = \{\pm 1, \pm i, \pm j, \pm k\}$ be the quaternion group. Then $Z = Z(Q) = \langle -1 \rangle = \{\pm 1\} \triangleleft Q$. The subgroups of Q containing Z form a sublattice [we use double lines to highlight this portion] which is isomorphic to the lattice of subgroups of Q/Z.

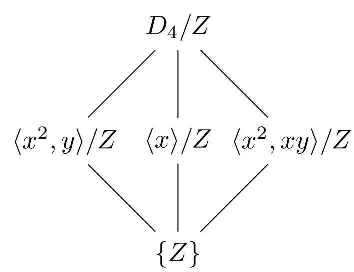




Note: The quaternion group is kind of a weirdo. It is a non-abelian group but all of it subgroups are normal subgroups. Also, since its not abelian, it's not cyclic. However, all of its proper subgroups (just exclude Q itself) are cyclic. Since quotients of cyclic groups are cyclic groups and quotients of normal subgroups are normal subgroups, all of the subgroups of Q/Z are cyclic (except Q/Z itself) and normal (in Q/Z).

Example: Let $D_4 = \langle x, y \mid x^4 = 1, y^2 = 1, (xy)^2 = 1 \rangle = \{1, x, x^2, x^3, y, xy, x^2y, x^3y\}$ be the dihedral group of order 8 (i.e., symmetries of a square). Then once again we will pick the center as our normal subgroup $Z = Z(D_4) = \langle x^2 \rangle = \{1, x^2\} \triangleleft D_4$. The subgroups of D_4 containing Z form a sublattice [we use double lines to highlight this portion] which is isomorphic to the lattice of subgroups of D_4/Z.





Note: $\langle x^2, y \rangle = \{1, x^2, y, x^2y\}$ and $\langle x^2, xy \rangle = \{1, x^2, xy, x^3y\}$. These two subgroups along with $\langle x \rangle$, the center $Z = \langle x^2 \rangle$, and the trivial subgroup $\{1\}$ are the normal subgroups of D_4. The normal subgroups lying above Z must map to normal subgroups of D_4/Z. Thus all subgroups of D_4/Z must be normal. This is true, but not very interesting. Notice that D_4/Z [like Q/Z above] is just the Klein 4-group (it's abelian – so of course all of its subgroups are normal).

One might see from the examples above how the structure of a quotient can inform us about a piece of the structure of the original group. However, it also reveals limitations. Notice that Q and D_4 are both non-abelian groups of order 8. Both of them have (cyclic) centers of order 2. Both of them have a quotient (by their center) isomorphic to the Klein 4-group. However, looking at the lattices to the left, they are quite different groups!