Math 3110 # Homework #1 **Due: Wed., Aug. 28th, 2024**

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PUT YOUR NAME ON YOUR WORK!

#1. Group axiom basics.

(a) We can equip $\mathbb{Z}_{\geq 0}$ (the non-negative integers) with the operation: $a \circ b = a^b$ (i.e., exponentiation). For example, $2 \circ 3 = 2^3 = 8$. In this context, discuss each of the following axioms: Closure, Associativity, (Right and Left) Identity, (Right and Left) Inverses, and Commutativity. If an axiom holds, explain why (=prove it). If an axiom fails, give a **concrete counterexample** showing it fails.

Example: If I wanted to show $\mathbb{Z}$ has a (two-sided) multiplicative identity, I would say, "Notice $1 \in \mathbb{Z}$ and for all $n \in \mathbb{Z}$, $1n = n = n1$." On the other hand, if I wanted to show $\mathbb{Z}$ fails to have multiplicative inverses, I would say, $\mathbb{Z}$ lack multiplicative inverses for some elements. In particular, 2 belongs to $\mathbb{Z}$ but its multiplicative inverse $2^{-1} = \frac{1}{2}$ does not."

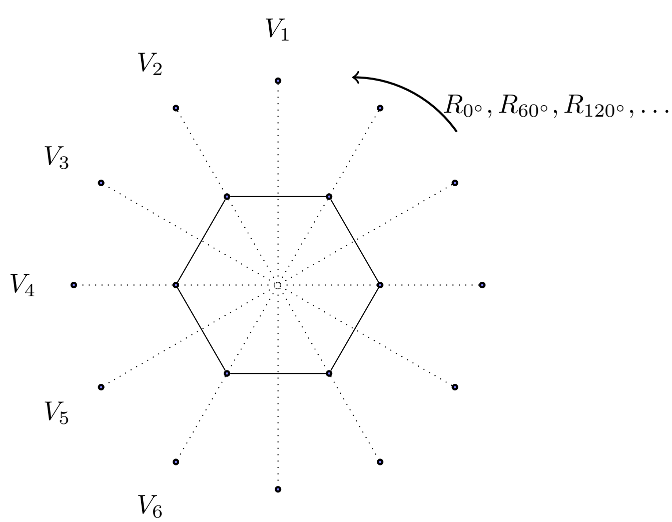
(b) Let's equip $\mathbb{Q}_{\neq 0}$ (the nonzero rational numbers) with a weird operation: $x \star y = \frac{2xy}{5}$.
For example, $-1 \star 5 = \frac{2(-1)5}{5} = -2$. As another example: $(x \star y) \star z = \left(\frac{2xy}{5}\right) \star z = \frac{2(2xy/5)z}{5} = \frac{4xyz}{25}$.
Obviously, $\mathbb{Q}_{\neq 0}$ is closed under $\star$. Show $\mathbb{Q}_{\neq 0}$ is a group under this operation.
You need to show $\star$ is associative, has an identity, and elements have inverses.
Also, is $\mathbb{Q}_{\neq 0}$ equipped with $\star$ an abelian group?

#2. Let G be a group with identity $e \in G$. In general, the law of exponents $(ab)^n = a^n b^n$ may fail to hold.

(a) Give a concrete example of a group G and elements $a, b \in G$ where $(ab)^{-1} \neq a^{-1}b^{-1}$.

(b) Prove G is an abelian group if and only if for all $a, b \in G$, $(ab)^{-1} = a^{-1}b^{-1}$.
Note: This is an "if and only if" statement. You need to prove two implications.

#3. Consider the dihedral group $D_6 = \{R_{0^\circ}, R_{60^\circ}, R_{120^\circ}, R_{180^\circ}, R_{240^\circ}, R_{300^\circ}, V_1, V_2, V_3, V_4, V_5, V_6\}$ (symmetries of a regular hexagon). [Rotations are done counter-clockwise and reflections are labeled in the picture below.]



(a) Compute $V_2 R_{60^\circ}$, $R_{180^\circ} V_5$, and $V_1 V_2$. [Draw some pictures!]

(b) Is D_6 Abelian? Why or why not?

(c) Make a table of inverses and orders for each element:

Element: $g =$	R_{0°	R_{60°	R_{120°	R_{180°	R_{240°	R_{300°	V_1	V_2	V_3	V_4	V_5	V_6
Inverse: $g^{-1} =$	???	$\cdots$										
Order: $\lvert g\rvert =$	???	$\cdots$										

Note: Recall that the order of an element g is the smallest positive power n such that g^n is the identity. If no such power exists the order of g is ∞.