Math 3110 # Homework #10 **Due: Mon., Nov. 25$^{\text{th}}$, 2024**

Please remember when submitting any work via email or in person to...

PUT YOUR NAME ON YOUR WORK!

#1 Let R be a commutative ring with 1 and let $a \in R$. Recall that $(a) = \{ra \mid r \in R\}$ is the principal ideal generated by a. In a commutative ring with 1, one says that $a, b \in R$ are **associates** if there exists some unit u (i.e., $u \in U(R)$) such that $a = ub$.

(a) What are the associates of $x^2 + 3x + 2$ in $\mathbb{Z}_5[x]$?
Note: The units of $\mathbb{Z}_5[x]$ are its nonzero constant polynomials.

(b) What are the associates of 55 in $\mathbb{R}$?

(c) Let R be an integral domain. Show that $(a) = (b)$ if and only if a and b are associates.

#2 Let $I = \{f(x) \in \mathbb{R}[x] \mid f(1) = 0 \text{ and } f'(1) = 0\}$. Prove that $I \triangleleft \mathbb{R}[x]$.

#3 Let R be a ring and let $I, J \triangleleft R$. One defines $I + J = \{x + y \mid x \in I \text{ and } y \in J\}$ to be the *sum* of I and J. Similarly, one defines $IJ = \{x_1y_1 + \cdots + x_\ell y_\ell \mid \text{for some } \ell \geq 0 \text{ and } x_1, \ldots, x_\ell \in I \text{ and } y_1, \ldots, y_\ell \in J\}$ to be the *product* of I and J. It can be shown that $I + J$, IJ, and $I \cap J$ are ideals of R.

(a) Prove that $I \cap J \triangleleft R$.

(b) Consider $I = (6)$ and $J = (15)$ in $\mathbb{Z}$. Calculate $I + J$, IJ, and $I \cap J$.

(c) Since $\mathbb{Z}$ is a principal ideal domain, $(a)(b) = (c)$, $(a) + (b) = (d)$, and $(a) \cap (b) = (\ell)$ for some $c, d, \ell \in \mathbb{Z}$. Make a conjecture about how a and b (where a and b are non-zero) are related to c, d, and ℓ.
Note: You don't have to prove your conjectures. Just tell me how you think this "ideal arithmetic" works.

#4 As a quick reminder, in $\mathbb{Z}$ and in $\mathbb{Z}_n$, we know that
subgroup = normal subgroup = cyclic subgroup = subring = ideal = principal ideal.

(a) Let R be a finite commutative ring with 1. Explain why ideals are prime if and only if they are maximal.

(b) Find all the ideals of $\mathbb{Z}_{98}$ and draw the corresponding lattice. Which ideals are prime? Which are maximal?

(c) Determine which ideals in $\mathbb{Z}$ are prime and which are maximal. [Prove your assertions.]
Note: Don't forget to consider the trivial ideal: $\{0\}$.