Math 3110 # Homework #6 **Due: For Practice**

Note: Due to the Hurricane. We will forego turning this assignment in for a grade. You need to know this material. It would be good to work through these problems yourselves, but I will just provide an answer key via email. **You do not need to turn this problem set in.**

#1 The following pairs of groups are **not** isomorphic. Prove it.

Hint/Note: Find a (group) property that does not match.

(a) $\mathbb{Z}^{3\times 3}$ and $\mathrm{GL}_3(\mathbb{Z})$

(b) $U(30)$ and $Q = \{\pm 1, \pm i, \pm j, \pm k\}$ (the quaternion group)

(c) S_5 and D_{60}

(d) $\mathbb{Q}$ and $\mathbb{Z}$
This one is a little tricky. *Hint:* If $0 \neq \frac{a}{b} \in \mathbb{Q}$, then $\frac{a}{2b} \notin \langle \frac{a}{b} \rangle$.

#2 The following pairs of groups are isomorphic. Prove it.

(a) $U(10)$ and $\mathbb{Z}_4$
Hint: You could cite a theorem and prove this in one line.

(b) $G = \left\{ \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \,\middle|\, a, b \in \mathbb{R} \right\}$ and $\mathbb{C}$

Notes:
- You may assume we already know G is a group (of 2×2 real matrices) under addition.
- You need to come up with an isomorphism, $\varphi : G \to \mathbb{C}$, and prove it is an isomorphism.

#3 Let $\varphi : G_1 \to G_2$ be an isomorphism

(a) Let $b \in G_1$ and $k \in \mathbb{Z}$. Explain why $x^k = b$ (in G_1) and $x^k = \varphi(b)$ (in G_2) must have the same number of solutions.

(b) Let K be a subgroup of G_2. Show that $\varphi^{-1}(K) = \{g \in G_1 \mid \varphi(g) \in K\}$ is a subgroup of G_1.

#4 Cayley's theorem tells us that $D_4 = \langle x, y \mid x^4 = 1, y^2 = 1, (xy)^2 = 1 \rangle = \{1, x, x^2, x^3, y, xy, x^2y, x^3y\}$ is isomorphic to a subgroup of S_8. Find such a subgroup using the ordering of elements: $1, x, x^2, x^3, y, xy, x^2y, x^3y$ (so, for example, x^3 is element #4 and x^2y is element #7). To help you get started, left multiplication by x sends 1 to x, x to x^2, x^2 to x^3, x^3 to 1, y to xy, etc. so it sends 1 to 2, 2 to 3, 3 to 4, 4 to 1, 5 to 6 etc. Thus x corresponds with (1234)(5678).

Note: Your answer should be a set of permutations: $\{(1), (1234)(5678), \ldots\}$