Math 3110 **Homework #7** **Due: Mon., Oct. 28th, 2024**

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#1 Suppose H and K are subgroups of some group G such that $|H| = 55$ and $|K| = 60$. Just considering divisibility criteria coming from Lagrange's Theorem, what is the largest size the subgroup $H \cap K$ could be? Suppose M contains both H and K. What is the smallest size M could be?

#2 Direct products.[1]

(a) Find the order of $(2, 12, j)$ in $U(15) \times \mathbb{Z}_{88} \times Q$ where $Q = \{\pm 1, \pm i, \pm j, \pm k\}$ is the quaternion group.

(b) Explain why $\mathbb{Z}_{99} \cong \mathbb{Z}_{11} \times \mathbb{Z}_9$ but $\mathbb{Z}_{99} \ncong \mathbb{Z}_3 \times \mathbb{Z}_{33}$.

In addition, list the distinct orders of elements in both $\mathbb{Z}_{11} \times \mathbb{Z}_9$ and $\mathbb{Z}_3 \times \mathbb{Z}_{33}$. Give an example of an element of each such order.

#3 Let $H = \{1, x^3, y, x^3y\} \subseteq D_6 = \{1, x, \ldots, x^5, y, xy, \ldots, x^5y\} = \langle x, y \mid x^6 = 1, y^2 = 1, xyxy = 1\rangle$. Make a table showing H is closed under D_6's operation (thus is a subgroup by the finite subgroup test).

What is $[D_6 : H]$ (i.e. the index of H in D_6)?

Find all of the left and right cosets of H in D_6. Is H a normal subgroup of D_6?

#4 Let G and H be groups. Show $G \times \{e\} = \{(g, e) \mid g \in G\}$ is a normal subgroup of $G \times H$ (where e is the identity of H).

Note: You need to show that $G \times \{e\}$ is a subgroup AND that it's normal.

#5 Let $D = \{(g, g) \mid g \in G\}$ where G is some group.

Note: D is the *diagonal* subgroup. It's all of the elements in $G \times G$ whose coordinates are equal.

(a) Show D is a subgroup of $G \times G$.

?Fun? Fact: D is not a normal subgroup of G when G is not abelian. Suppose $a, b \in G$ and $ab \neq ba$. Then $aba^{-1} \neq b$. Notice $(b, b) \in D$, but $(a, b)(b, b)(a, b)^{-1} = (a, b)(b, b)(a^{-1}, b^{-1}) = (aba^{-1}, bbb^{-1}) = (aba^{-1}, b) \notin D$ since $aba^{-1} \neq b$.

(b) Show $D \cong G$.

[1] Gallian uses $G \oplus H$ for direct products. I will use the more standard $G \times H$ notation.