Math 3110 # Homework #8 **Due: Mon., Nov. 4th, 2024**

Please remember when submitting any work via email or in person to...

PUT YOUR NAME ON YOUR WORK!

#1 Let $H = \langle x^4 \rangle = \{1, x^4\} \subseteq D_8 = \langle x, y \mid x^8 = 1, y^2 = 1, (xy)^2 = 1 \rangle = \{1, x, \dots, x^7, y, xy, \dots, x^7 y\}$.

(a) Compute the (distinct) left and right cosets of H to confirm this is a normal subgroup of D_8.

(b) Write down a Cayley table for D_8/H.

(c) Is D_8/H abelian? Is it cyclic? Explain you answers.

#2 Quotients in $\mathbb{Z}_n$.

(a) Let $H = \langle 20 \rangle \subseteq \mathbb{Z}_{24}$.

i. List the elements of H (i.e., $H = \{???\}$) and explain why it is a normal subgroup of $\mathbb{Z}_{24}$.

ii. Compute the (distinct) cosets of H in $\mathbb{Z}_{24}$.

iii. Write down a Cayley table for $\mathbb{Z}_{24}/H$. What familiar group is this isomorphic to?

(b) Consider $k, \ell, n \in \mathbb{Z}_{>0}$ where $n = k\ell$. We attempt to define a function $\varphi : \mathbb{Z}_n \to \mathbb{Z}_\ell$ by $\varphi(x) = x$.
[Using more precise notation: $\varphi(x + n\mathbb{Z}) = x + \ell\mathbb{Z}$.]

Prove that φ is a *well-defined* homomorphism.

Determine φ's kernel and image. Then apply the first isomorphism theorem.

#3 Consider $\varphi : U(9) \to U(9)$ defined by $\varphi(x) = x^2$. Show φ is a homomorphism. Find its kernel and range. Consider $U(9)/\mathrm{Ker}(\varphi)$. Is this quotient abelian? Is this quotient cyclic?

#4 Let $H \lhd G$ where G is a finite group. Suppose $xH \in G/H$ and xH is an element of order 4 in G/H. Show that G must have an element of order 4.

Warning: Just because xH has order 4 (in G/H) does not mean x has order 4 in G.
So x itself might not work, but the right *power* of x will.