Appendix #22 Negate the following: "For every real number r, the square of r is nonnegative."

Answer: "There exists a real number r such that r is negative." ...or less words and more symbols... "$\exists\, r \in \mathbb{R}$ s.t. $r^2 < 0$."

[Of course, our answer is a false statement since $r^2 \geq 0 \;\forall\, r \in \mathbb{R}$.]

1.1 #14 Let $C = \{x \,|\, x = 3r - 1 \text{ for some } r \in \mathbb{Z}\}$ and $D = \{x \,|\, x = 3s + 2 \text{ for some } s \in \mathbb{Z}\}$. Show that $C = D$.

proof: Suppose that $x \in C \Rightarrow \exists\, r \in \mathbb{Z}$ s.t. $x = 3r - 1$. But $x = 3r - 1 = 3(r-1) + 2$ and $r - 1 \in \mathbb{Z}$ since $r \in \mathbb{Z}$. Thus (setting $s = r - 1$) we get that $x = 3s + 2$ where $s \in \mathbb{Z}$ and thus $x \in D \therefore C \subseteq D$.

Next, suppose that $x \in D \Rightarrow \exists\, s \in \mathbb{Z}$ s.t. $x = 3s+2$. But $x = 3s+2 = 3(s+1)-1$ and $s+1 \in \mathbb{Z}$ since $s \in \mathbb{Z}$. Thus (setting $r = s + 1$) we get that $x = 3r - 1$ where $r \in \mathbb{Z}$ and thus $x \in C \therefore D \subseteq C$.

Finally, since $C \subseteq D$ and $D \subseteq C$, we can conclude that $C = D$.

1.1 #30 Show that $A \cup B = A \cup (B - A)$.

proof: Suppose that $x \in A \cup (B - A) \Rightarrow x \in A$ or $x \in B - A \Rightarrow x \in A$ or ($x \in B$ and $x \notin A$) $\Rightarrow x \in A$ or $x \in B \Rightarrow x \in A \cup B \therefore A \cup (B - A) \subseteq A \cup B$.

Suppose that $x \in A \cup B$. Notice that either $x \in A$ or $x \notin A$. Consider two cases:

- $x \in A \Rightarrow x \in A \cup C$ where C could be anything – like, for example, $C = B - A \therefore x \in A \cup (B - A)$.
- $x \notin A \Rightarrow x \in B$ since $x \in A \cup B$ (so $x \in A$ or $x \in B$) and $x \notin A$. So we have that $x \in B$ and $x \notin A \Rightarrow x \in B - A$ and so $x \in C \cup (B - A)$ for any set C – like, for example, $C = A$. $\therefore x \in A \cup (B - A)$.

$\therefore A \cup B \subseteq A \cup (B - A)$.

Finally, since $A \cup B \subseteq A \cup (B - A)$ and $A \cup (B - A) \subseteq A \cup B$, we can conclude that $A \cup B = A \cup (B - A)$.

1.2 #4a,c,d,i Let $f : \mathbb{Z} \to \mathbb{Z}$ (both the domain and codomain of f is $\mathbb{Z}$). For each map, prove or disprove: f is one-to-one and prove or disprove: f is onto.

#4a $f(x) = 2x$:

f is one-to-one – proof: suppose that $f(x) = f(y) \Rightarrow 2x = 2y \Rightarrow x = y$. Thus f is injective.

f is not onto – proof: suppose that $f(x) = 1$ then $2x = 1$ and thus $x = 1/2$. But $1/2 \notin \mathbb{Z} \therefore 1$ is not in the image of f (so f is not onto).

#4c $f(x) = x + 3$:

f is one-to-one – proof: suppose that $f(x) = f(y) \Rightarrow x + 3 = y + 3 \Rightarrow x = y$. Thus f is injective.

f is onto – proof: suppose that $y \in \mathbb{Z}$ then $y - 3 \in \mathbb{Z}$ and $f(y - 3) = (y - 3) + 3 = y$. Thus y is in the image of f. Therefore, the image of f contains all possible integers and thus f is onto.

Alternate proof: Let $g : \mathbb{Z} \to \mathbb{Z}$ be defined by $g(x) = x - 3$. Notice that $(f \circ g)(x) = f(g(x)) = f(x - 3) = x$ and $(g \circ f)(x) = g(f(x)) = g(x + 3) = x$. Therefore, $f \circ g = id_{\mathbb{Z}} = g \circ f$. Thus f is invertible (with inverse $f^{-1} = g$) and so f is bijective (both one-to-one and onto).

#4d $f(x) = x^3$:

f is one-to-one – proof: suppose that $f(x) = f(y) \Rightarrow x^3 = y^3 \Rightarrow x = y$. Thus f is injective.

f is not onto – proof: suppose that $f(x) = 2 \Rightarrow x^3 = 2 \Rightarrow x = \sqrt[3]{2} \notin \mathbb{Z} \therefore 2$ is not in the image of f.

#4i $f(x) = \begin{cases} x & x \text{ is even} \\ \frac{x-1}{2} & x \text{ is odd} \end{cases}$:

f is not one-to-one – proof: $f(1) = (1 - 1)/2 = 0 = f(0)$.

f is onto – proof: suppose that $y \in \mathbb{Z}$. Then $2y + 1 \in \mathbb{Z}$ and $2y + 1$ is odd $\therefore f(2y + 1) = ((2y + 1) - 1)/2 = y$. Thus every integer is in the range of $f \therefore f$ is surjective.

1.2 #22 Let $f : A \to B$ and $A, B \neq \emptyset$. Let $S_1, S_2 \subseteq A$.

(a) Show $f(S_1 \cup S_2) = f(S_1) \cup f(S_2)$.

proof: Let $y \in f(S_1 \cup S_2) \Rightarrow \exists\, x \in S_1 \cup S_2$ s.t. $f(x) = y \Rightarrow y = f(x)$ where $x \in S_1$ or $x \in S_2$. If $x \in S_1$, then $y = f(x) \in f(S_1)$ and if $x \in S_2$, then $y = f(x) \in f(S_2) \therefore y \in f(S_1) \cup f(S_2)$ $\therefore f(S_1 \cup S_2) \subseteq f(S_1) \cup f(S_2)$.

Now suppose that $y \in f(S_1) \cup f(S_2) \Rightarrow y \in f(S_1)$ or $y \in f(S_2)$. If $y \in f(S_1)$, then $\exists\, x \in S_1$ s.t. $f(x) = y$ and if $y \in f(S_2)$, then $\exists\, x \in S_2$ s.t. $f(x) = y$. Therefore, $\exists\, x \in S_1$ or $x \in S_2$ s.t. $f(x) = y \therefore \exists\, x \in S_1 \cup S_2$ s.t. $f(x) = y$. Thus $y \in f(S_1 \cup S_2) \therefore f(S_1) \cup f(S_2) \subseteq f(S_1 \cup S_2)$.

Finally, we conclude that $f(S_1) \cup f(S_2) = f(S_1 \cup S_2)$.

(b) Show $f(S_1 \cap S_2) \subseteq f(S_1) \cap f(S_2)$

proof: Let $y \in f(S_1 \cap S_2) \Rightarrow \exists\, x \in S_1 \cap S_2$ s.t. $f(x) = y$. Thus $x \in S_1$ and $x \in S_2$ and $f(x) = y$. So $y \in f(S_1)$ and $y \in f(S_2) \therefore y \in f(S_1) \cap f(S_2)$. Thus $f(S_1 \cap S_2) \subseteq f(S_1) \cap f(S_2)$.

(c) Give an example where $f(S_1 \cap S_2) \neq f(S_1) \cap f(S_2)$.

Let $A = B = \{1, 2\}$ and $f : A \to B$ where $f(1) = f(2) = 1$. In addition, let $S_1 = \{1\}$ and $S_2 = \{2\}$. Then, $f(S_1 \cap S_2) = f(\emptyset) = \emptyset$ but $f(S_1) \cap f(S_2) = \{1\} \cap \{1\} = \{1\} \therefore$ $f(S_1 \cap S_2) \neq f(S_1) \cap f(S_2)$.

Note: Any map which is not one-to-one will furnish us with an example. In fact, we can prove the following statement:

Let $f : A \to B$ be a map. Then f is injective if and only if for all $S_1, S_2 \subseteq A$ we have that $f(S_1 \cap S_2) = f(S_1) \cap f(S_2)$.

proof: Suppose that f is injective and $S_1, S_2 \subseteq A$. We already know from part (b) that $f(S_1 \cap S_2) \subseteq f(S_1) \cap f(S_2)$. We need to show the reverse inclusion. Suppose that $y \in f(S_1) \cap f(S_2) \Rightarrow y \in f(S_1)$ and $y \in f(S_2) \Rightarrow \exists\, x_1 \in S_1$ and $x_2 \in S_2$ s.t. $y = f(x_1) = f(x_2)$. But f is injective $\therefore x_1 = x_2$. So x_1 is in both S_1 and S_2. Thus $x_1 \in S_1 \cap S_2$ and so $y \in f(S_1 \cap S_2) \therefore f(S_1) \cap f(S_2) \subseteq f(S_1 \cap S_2)$. So we have shown that f injective implies that $f(S_1) \cap f(S_2) = f(S_1 \cap S_2)$ ($\forall\, S_1, S_2 \subseteq A$).

Now suppose that f is not injective. We need to show that $\exists\, S_1, S_2 \subseteq A$ s.t. $f(S_1 \cap S_2) \neq f(S_1) \cap f(S_2)$. Well, f is not injective $\therefore \exists\, x_1, x_2 \in A$, $x_1 \neq x_2$ s.t. $f(x_1) = f(x_2)$ (for convenience set $y = f(x_1) = f(x_2)$). Now consider $S_1 = \{x_1\}$ and $S_2 = \{x_2\}$. Then $f(S_1 \cap S_2) = f(\emptyset) = \emptyset$ and $f(S_1) \cap f(S_2) = \{y\} \cap \{y\} = \{y\}$. Therefore, $f(S_1 \cap S_2) \neq f(S_1) \cap f(S_2)$. So we have shown that if f fails to be injective, then f will fail to preserve set intersections.

(d) Show that $f(S_1) - f(S_2) \subseteq f(S_1 - S_2)$.

proof: Suppose that $y \in f(S_1) - f(S_2) \Rightarrow y \in f(S_1)$ and $y \notin f(S_2) \therefore \exists\, x \in S_1$ s.t. $f(x) = y$. Notice that if $x \in S_2$ then $y = f(x) \in f(S_2) \therefore x \notin S_2$ and so $x \in S_1 - S_2$ and thus $y = f(x) \in f(S_1 - S_2) \therefore f(S_1) - f(S_2) \subseteq f(S_1 - S_2)$.

(e) Give an example where $f(S_1) - f(S_2) \neq f(S_1 - S_2)$.

Again consider $A = B = \{1, 2\}$ and $f : A \to B$ defined by $f(1) = f(2) = 1$. Let $S_1 = A$ and $S_2 = \{2\}$. Then $f(S_1) - f(S_2) = \{1\} - \{1\} = \emptyset$ and $f(S_1 - S_2) = f(\{1, 2\} - \{2\}) = f(\{1\}) = \{1\} \therefore f(S_1) - f(S_2) \neq f(S_1 - S_2)$.

Note: Again, any map which is not one-to-one will furnish us with an example. In fact, we can prove the following statement:

Let $f : A \to B$ be a map. Then f is injective if and only if for all $S_1, S_2 \subseteq A$ we have that $f(S_1 - S_2) = f(S_1) - f(S_2)$.

proof: Suppose that f is injective and $S_1, S_2 \subseteq A$. We already know from part (d) that $f(S_1) - f(S_2) \subseteq f(S_1 - S_2)$. We need to show the reverse inclusion. Suppose that $y \in f(S_1 - S_2) \Rightarrow \exists\, x \in S_1 - S_2$ s.t. $y = f(x)$. In particular, $x \in S_1$ and so $y = f(x) \in f(S_1)$ Now suppose $y \in f(S_2) \Rightarrow \exists\, z \in S_2$ s.t. $y = f(z)$. But f is injective $\therefore$ since $f(x) = f(z)$ we have that $x = z$ and so $x \in S_2$. This is impossible since $x \in S_1 - S_2$ (and thus $x \notin S_2$). Therefore, $y \notin f(S_2)$ and thus $y \in f(S_1) - f(S_2) \therefore f(S_1 - S_2) \subseteq f(S_1) - f(S_2)$. So we have shown that f injective implies that $f(S_1 - S_2) = f(S_1) - f(S_2)$ ($\forall\, S_1, S_2 \subseteq A$).

Now suppose that f is not injective. We need to show that $\exists\, S_1, S_2 \subseteq A$ s.t. $f(S_1 - S_2) \neq f(S_1) - f(S_2)$. Well, f is not injective $\therefore \exists\, x_1, x_2 \in A$, $x_1 \neq x_2$ s.t. $f(x_1) = f(x_2)$ (for convenience set $y = f(x_1) = f(x_2)$). Now consider $S_1 = \{x_1, x_2\}$ and $S_2 = \{x_2\}$. Then $f(S_1) - f(S_2) = \{y\} - \{y\} = \emptyset$ and $f(S_1 - S_2) = f(\{x_1, x_2\} - \{x_2\}) = f(\{x_1\}) = \{y\}$. Therefore, $f(S_1 - S_2) \neq f(S_1) - f(S_2)$. So we have shown that if f fails to be injective, then f will fail to preserve set differences.