Let $S \subseteq T$ such that $S \neq \phi$ (S is non-empty).
Binary Relation: Let $* : S \times S \to S$ be a map denoted by $a * b$ for all $a, b \in S$. Such a map $*$ is called a *binary operation* or *binary relation* on S. Notice that part of the definition of a binary relation is that the range of $*$ is contained in S. Thus if we have $* : S \times S \to T$, then to check that $*$ is a binary relation we must verify that $a * b \in S$ for all $a, b \in S$ (this is called checking *closure*).

Let S be a non-empty set with a binary operation $*$ (so S is *closed* under the operation $*$).
Associativity: If $a * (b * c) = (a * b) * c$ for all $a, b, c \in S$, then we say $*$ is an *associative* operation on S.

Identity: Suppose we have an element $e \in S$ such that $a * e = a = e * a$ for all $a \in S$. Then e is called an *identity* element for the operation $*$ on S.

Inverses: Suppose that $e \in S$ is an identity element. Then S has *inverses* if for each $a \in S$ there exists some $b \in S$ such that $a * b = e = b * a$ (b is the *inverse* of a).

Commutativity: If $a * b = b * a$ for all $a, b \in S$, then $*$ is a *commutative* operation on S.

Some basic algebraic objects...
Semigroup = Closure + Associativity: A non-empty set with an associative binary operation is called a *semigroup*. *Note:* Some authors assume their semigroups have an identity (for them semigroup = monoid).

Monoid = Closure + Associativity + Identity: A semigroup with an identity element is called a *monoid*.

Group = Closure + Associativity + Identity + Inverses: A monoid such that each element has an inverse is called a *group*.

Abelian Group = Group + Commutativity: A group with a commutative binary operation is called an *Abelian group* (or sometimes a commutative group).

Ring: Let R be a non-empty set equipped with two binary operations:
Closure under Addition: $+ : R \times R \to R$ called *addition* denoted $a + b$ for all $a, b \in R$ and
Closure under Multiplication: $\cdot : R \times R \to R$ called *multiplication* denoted ab for all $a, b \in R$.
Then R is a *ring* if the following axioms hold:

(i) R paired with the operation $+$ is an Abelian group – denote the identity element by 0. That is:
Addition is associative: For all $a, b, c \in R$ we have $a + (b + c) = (a + b) + c$.
Additive identity: There exists some $0 \in R$ such that for all $a \in R$ we have $a + 0 = a = 0 + a$.
Additive inverses: For all $a \in R$ there exists $-a \in R$ such that $a + (-a) = 0 = (-a) + a$.
Addition is commutative: For all $a, b \in R$ we have $a + b = b + a$.

(ii) R paired with the multiplicative operation is a semigroup. That is:
Multiplication is associative: For all $a, b, c \in R$ we have $a(bc) = (ab)c$.

(iii) The multiplication on R distributes across the addition on R. That is for all $a, b, c \in R$:
Left-Distributivity: For all $a, b, c \in R$ we have $a(b + c) = ab + ac$.
Right-Distributivity: For all $a, b, c \in R$ we have $(a + b)c = ac + bc$.

For all $a, b \in R$, let $L_a : R \to R$ be defined by $L_a(b) = ab$ and $R_a : R \to R$ by $R_a(b) = ba$. These are left and right multiplication operators. Notice that the distributive laws say (for every $a, b, c \in R$) $L_a(b+c) = a(b+c) = ab+ac = L_a(b) + L_a(c)$ and $R_c(a + b) = (a + b)c = ac + bc = R_c(a) + R_c(b)$. In other words, the distributive laws merely say that left and right multiplications are Abelian group homomorphisms (with respect to addition). So a ring $(R, +, \cdot)$ is an Abelian group $(R, +)$ and semigroup $(R, \cdot)$ such that left and right multiplication operators are homomorphisms with respect to the $(R, +)$ structure. *Note:* A homomorphism sends an identity to identity. Thus $L_a(0) = 0 = R_a(0)$, so $a0 = 0 = 0a$ for all $a \in R$. It also preserves inverses so that $L_a(-b) = -L_a(b)$ and $R_b(-a) = -R_b(a)$. Thus $a(-b) = -(ab) = b(-a)$ for all $a, b \in R$.

Note of possible interest: If R has a multiplicative identity, requiring addition to be commutative is redundant! Why? Suppose $a, b \in R$. Then $-(a + b) = (-b) + (-a)$ using the socks-shoes inverse property. [This property is true in any system where inverses make sense.] Next, consider $(-1)c = R_c(-1) = -R_c(1) = -(1c) = -c$ using the fact that right distributivity implies right multiplications are homomorphisms and thus preserve inverses. Thus $-(a+b) = -(1(a+b)) = (-1)(a+b) = (-1)a+(-1)b = (-a)+(-b)$ where we just used the fact that 1 is the identity ($1(a + b) = a + b$), the negation property from above ($-(1(a + b)) = (-1)(a + b)$, $(-1)a = -a$, and $(-1)b = -b$), and the left distributive law ($(-1)(a + b) = (-1)a + (-1)b$). Therefore, $(-b) + (-a) = -(a + b) = (-a) + (-b)$. Now add $a + b$ on the left of both sides of the equation and $b + a$ on the right of both sides and get $b + a = a + b$.

Let R be a ring.

Zero Divisors: Let $a, b \in R$ be two non-zero elements ($a \neq 0$ and $b \neq 0$). Then if $ab = 0$, we call both a and b *zero divisors*. More precisely, a is a *left* zero divisor and b is a *right* zero divisor.

Units: Let $a \in R$. If there exists $b \in R$ such that $ab = 1 = ba$, then a is called a *unit* in R. The collection of all units of R is called the *group of units* and is denoted $U(R)$ or better yet $R^{\times}$.

For example, recall $U(n) = U(\mathbb{Z}_n)$ are the units (elements with multiplicative inverses) in $\mathbb{Z}_n$.

Again, let R be a ring. Special types of rings...

Ring with Identity: If there exists some element $1 \in R$ such that $a1 = a = 1a$ for all $a \in R$, then R is called a *ring with identity* (or ring with 1 or ring with unity).

Commutative Ring: If the multiplication on R is commutative (that is $ab = ba$ for all $a, b \in R$), then R is called a *commutative ring*.

Integral Domain: Let R be a commutative ring with identity such that $1 \neq 0$. If R has no zero divisors, then R is an *integral domain*. This means that for all $a, b \in R$ if $ab = 0$, then either $a = 0$ or $b = 0$.

Field: Let R be a commutative ring with identity such that $1 \neq 0$. If every non-zero element of R is a unit, then R is a *field*. That means that for all $a \in R$ there exists $a^{-1} \in R$ such that $aa^{-1} = 1 = a^{-1}a$.

Domain: If we remove the assumption of commutativity from the definition of an integral domain, we get the definition of a *domain*.

Division Ring: If we remove the assumption of commutativity from the definition of a field, we get the definition of a *division ring* (or *skew field*).

Some notation...

Additive Notation: Typically the "+" symbol is only used for commutative operations, and the identity element is denoted by "0". Let's say that $(R, +)$ forms an Abelian group (this is true for any ring R). Then each element $a \in R$ has a *unique* additive inverse which we denote by $-a$. Let $n \in \mathbb{Z}_{>0}$ then by na we mean: $na = \underbrace{a + a + \cdots + a}_{n \text{ times}}$. Also, $0a$ is defined to be $0a = 0$. Notice that the zero on the left hand side is the integer zero whereas the zero on the right hand side is the zero of the group (or ring). Since $-a$ exists, we define: $(-n)a = \underbrace{(-a) + (-a) + \cdots + (-a)}_{n \text{ times}}$.

Various laws of exponents hold: For any $m, n \in \mathbb{Z}$ and $a, b \in R$, we have $(n+m)a = na + ma$, $n(ma) = (nm)a$, and (since addition is commutative) $n(a+b) = na+nb$. *Note:* This last law of exponents looks like a distributive law, but it is not. For example: $2(a+b) = (a+b)+(a+b) = (a+a)+(b+b) = 2a+2b$ (using commutativity and associativity). Sometimes notation is ambiguous. For example, $0a$ could be the ring's zero element times a or it could be the zero-th additive power of a. Either way, this results in the ring's additive identity 0: $0a = 0$. Likewise, $1a$ could be the first additive power of a or if R has 1, this could mean 1 times a. Either way, we get $1a = a$. Thus these ambiguities don't typically matter.

Multiplicative Notation: Typically the multiplication in a ring is denoted by juxtaposition (putting symbols next to each other). If a ring has a multiplicative element, it is usually denoted by "1". If R is a ring with 1 and $a \in R$, then a may or may not have a (multiplicative) inverse. However, if a does have an inverse, this inverse is *unique* and is denoted by a^{-1}. Let $n \in \mathbb{Z}_{>0}$ and $a \in R$ (a ring), then by a^n we mean: $a^n = \underbrace{aa \ldots a}_{n \text{ times}}$. If R is a ring with 1, we define $a^0 = 1$ where the zero in the exponent is the integer zero and the 1 on the right hand side is the multiplicative identity of R. Finally, if R is a ring with 1 and a is a unit of R (i.e., it has a multiplicative inverse), then we define $a^{-n} = \underbrace{a^{-1}a^{-1} \ldots a^{-1}}_{n \text{ times}}$.

Again, we have laws of exponents: For any m, n that make sense, $a^{m+n} = a^m a^n$ and $(a^m)^n = a^{mn}$. On the other hand, $(ab)^n = a^n b^n$ is only guaranteed to hold if a and b commute. For example, if a and b are units (i.e., have multiplicative inverses), $(ab)^{-1} = b^{-1}a^{-1}$ (which may or may not be equal to $a^{-1}b^{-1}$).

WARNING: Some (in fact many) authors require that **all** rings have multiplicative identities. In fact, what we call a ring they call a *rng* (the i has been deleted) [Pronounced "rung"]. Also, some (very odd misguided) authors require that in addition that $1 \neq 0$. For such authors, the zero ring $R = \{0\}$ is not a ring!