$\mathbb{Z}$ = integers, $\mathbb{E}$ = even integers, $\mathbb{Q}$ = rational numbers, $\mathbb{R}$ = real numbers, $\mathbb{C}$ = complex numbers, and $\mathbb{Z}_n$ = integers (mod n) are all rings under the usual additions and multiplications.

$\mathbb{Z}[i] = \{a + bi \,|\, a, b \in \mathbb{Z}\}$ are the *Gaussian integers*. Addition and multiplication are defined as follows: $(a+bi)+(c+di) = (a+c)+(b+d)i$ and $(a+bi)(c+di) = (ac-bd)+(ad+bc)i$ so that $i^2 = -1$ (i.e., $i = \sqrt{-1}$). $\mathbb{Z}[i]$ is a subring of the complex numbers.

Ring	Has Identity	Commutative	Integral Domain	Field	Characteristic	Finite
$\{0\}$	√	√			1	√
$\mathbb{Z}$	√	√	√		0	
$E = 2\mathbb{Z}$		√			0	
$\mathbb{Q}$	√	√	√	√	0	
$\mathbb{R}$	√	√	√	√	0	
$\mathbb{C}$	√	√	√	√	0	
$\mathbb{Z}_n$	√	√	only if n is prime	only if n is prime	n	√
$\mathbb{Z}[i]$	√	√	√		0	

$\mathbb{H} = \{a+bi+cj+dk \,|\, a,b,c,d \in \mathbb{R}\}$ are called the *Quaternions* (the "$\mathbb{H}$" is in honor of their discoverer William Hamilton). Addition is defined as follows: $(a_1+b_1i+c_1j+d_1k)+(a_2+b_2i+c_2j+d_2k) = (a_1+a_2)+(b_1+b_2)i+(c_1+c_2)j+(d_1+d_2)k$. To multiply use distributive laws, multiply real numbers as usual, and multiply i, j, k as with the quaternion group ($ij = k$, $ji = -k$, $j^2 = -1$ etc. Briefly: $i^2 = j^2 = k^2 = ijk = -1$). They are an infinity non-commutative ring with identity. Their characteristic is 0. They are neither an integral domain nor a field since they are not commutative. However, $\mathbb{H}$ has no zero divisors (they are a *domain* and every non-zero element has a multiplicative inverse (they are a *skew-field* or *division ring*).

Building ring out of things:

- Let G be any Abelian group with operation + and identity 0. Give G the "zero multiplication" defined by $ab = 0$ for all $a, b \in G$. Then G becomes a commutative ring. Every non-zero element of G is a zero divisor. If $G \neq \{0\}$, then G has no (multiplicative) identity.
- Let R_1 and R_2 be rings. Form the product ring $R_1 \times R_2$ by defining: $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$ and $(x_1, y_1)(x_2, y_2) = (x_1x_2, y_1y_2)$ for all $x_1, x_2 \in R_1$ and $y_1, y_2 \in R_2$. Notice that $R_1 \times R_2$ is commutative if and only if both R_1 and R_2 are commutative. Also, $R_1 \times R_2$ has an identity if and only if both R_1 and R_2 have identities. Suppose that $R_1 \neq \{0\}$ and $R_2 \neq \{0\}$. Let $0 \neq a \in R_1$ and $0 \neq b \in R_2$. Then $(a, 0)(0, b) = (a0, 0b) = (0, 0)$, so $R_1 \times R_2$ always has zero divisors (if R_1 and R_2 aren't trivial) and thus cannot be an integral domain or a field. *Note:* $|R_1 \times R_2| = |R_1| \cdot |R_2|$. So, for example, $|\mathbb{Z}_2 \times \mathbb{Z}_3| = 2 \cdot 3 = 6$.
- Let R be a ring. $M_n(R) = R^{n\times n}$ is the ring of $n \times n$ matrices with entries in R. Remember if the ij-entries of A and B are a_{ij} and b_{ij} respectively, then the ij-entry of $A + B$ is just $a_{ij} + b_{ij}$ and the ij-entry of AB is $\sum_{k=1}^{n} a_{ik}b_{kj}$. One can show that $R^{n\times n}$ has an identity if and only if R has an identity. Also, $R^{n\times n}$ is **not** commutative except in the case that (1) R is commutative and $n = 1$ or (2) R has the zero mutiplication (that is $ab = 0$ for all $a, b \in R$). *Note:* $|R^{n\times n}| = |R|^{n^2}$. So, for example, $|(\mathbb{Z}_3)^{2\times 2}| = 3^{2^2} = 3^4 = 81$.
- Let R be a ring. $R[[x]] = \{a_0 + a_1x + a_2x^2 + ... \,|\, a_i \in R\}$ is the ring of formal power series with coefficients in R. If $f(x) = a_0 + a_1x + \dots$ and $g(x) = b_0 + b_1x + \dots$, then $f(x) + g(x) = (a_0 + b_0) + (a_1 + b_1)x + ...$ and $f(x)g(x) = a_0b_0 + (a_0b_1 + a_1b_0)x + \cdots + (\sum_{i=0}^{k} a_ib_{k-i})x^k + \dots$. It is easy to show that $R[[x]]$ is commutative if and only if R is commutative. Also, $R[[x]]$ has a multiplicative identity if and only if R has 1. In fact, R is an integral domain if and only if $R[[x]]$ is an integral domain.
- Let R be a ring. $R[x] = \{a_0 + a_1x + \cdots + a_nx^n \,|\, a_i \in R\}$ is a subring of $R[[x]]$ called the ring of polynomials with coefficients in R. Again, $R[x]$ is commutative if and only if R is commutative; $R[x]$ has a multiplicative identity if and only if R has 1; and R is an integral domain if and only if $R[x]$ is an integral domain.
  For example, $\mathbb{Z}_5[x]$ are polynomials with coefficients in $\mathbb{Z}_5$. This is an infinite ring with characteristic 5.