

We say a permutation is even if it can be written as a product of an even number of (usually non-disjoint) transpositions (i.e. 2-cycles). Likewise a permutation is odd if it can be written as a product of an odd number of transpositions. The first question is, “Can any permutation be written as a product of transpositions?” The answer is “Yes.” ...well if we’re working in S_n for $n > 1$ (of course, S_1 doesn’t have any transpositions...it just has the identity). For the remainder of this handout, fix some $n > 1$. Recall the trick:

$$(a_1 a_2 \dots a_\ell) = (a_1 a_\ell)(a_1 a_{\ell-1}) \cdots (a_1 a_3)(a_1 a_2)$$

Also, $(1) = (12)(12)$. Therefore, any cycle of any length can be written as a product of transpositions. Now since every permutation can be written as a product of (disjoint) cycles, we can use this trick on each cycle and get: *Every permutation can be written as a product of transpositions.*

Consider the example:

$$(123) = (13)(12) = (34)(14)(34)(12) = (34)(12)(12)(14)(34)(12) = \dots$$

Notice that (123) can be written as a product of transpositions in (infinitely) many different ways. However, the three ways shown above have 2, 4, and 6 transpositions respectively – thus (123) is even. Our next question is, “It is possible that (123) or any other permutation is both even and odd?” The answer is “No” but this requires some proof.

Lemma: The identity is even — and not odd.

proof: First, we know that $(1) = (12)(12)$, so the identity is even. Now suppose that $(1) = (a_1 a_2) \cdots (a_{\ell-1} a_\ell)$. We want to show that there **must** be an even number of these transpositions. First, let’s see how to push transpositions past each other. There are 4 cases of interest: Let a, b, c, d be distinct elements of the set $\{1, 2, \dots, n\}$.

- $(cd)(ab) = (ab)(cd)$ — disjoint cycles commute.
- $(bc)(ab) = (acb) = (cba) = (ca)(cb)$ — multiply out, cyclicly permute, transposition trick.
- $(ac)(ab) = (abc) = (bca) = (ba)(bc)$ — same as before.
- $(ab)(ab) = (1)$

Notice that in the first 3 cases, we can move a to the left. In the last case, we cancel a out completely.

Now suppose a is the largest number appearing among all the transpositions in $(a_1 a_2) \cdots (a_{\ell-1} a_\ell)$. We can take the right-most occurrence of a and move it to the left. As we move all of the a ’s to the left, at some point, the a ’s must cancel out (we have to end up with the “ $(ab)(ab)$ ” case). If not, we would have $(1) = (ab)\tau$ with no a ’s appearing in τ . But this is impossible since τ maps a to a (no occurrences of a in τ) and (ab) maps a to b so that $(ab)\tau$ is not the identity! Therefore, we can get rid of all of the occurrences of a by canceling out transpositions in **pairs**. Continuing in this fashion (after a is gone pick the next smallest remaining number), we will eventually cancel out all of the transpositions. Since cancelations always occur in pairs, it must be that (1) was written as an even number of transpositions. Therefore, (1) cannot be odd. $\square$

Theorem: Every permutation in S_n ($n > 1$) is either even or odd, but not both.

proof: Let $\sigma \in S_n$. We know by the transposition trick above that σ can be written as a product of transpositions. Suppose $\sigma = (a_1 a_2) \cdots (a_{2\ell-1} a_{2\ell}) = (b_1 b_2) \cdots (b_{2k-1} b_{2k})$. Then

$$\begin{aligned} (1) = \sigma \sigma^{-1} &= (a_1 a_2) \cdots (a_{2\ell-1} a_{2\ell}) [(b_1 b_2) \cdots (b_{2k-1} b_{2k})]^{-1} \\ &= (a_1 a_2) \cdots (a_{2\ell-1} a_{2\ell}) (b_{2k-1} b_{2k})^{-1} \cdots (b_1 b_2)^{-1} \\ &= (a_1 a_2) \cdots (a_{2\ell-1} a_{2\ell}) (b_{2k-1} b_{2k}) \cdots (b_1 b_2) \end{aligned}$$

So we have written (1) as the product of $\ell + k$ transpositions. Our lemma says that $\ell + k$ must be even. Therefore, either both k and ℓ are even or both are odd. $\square$

QUICK COMPUTATIONS:

We can quickly determine whether a permutation is even or odd by looking at its cycle structure. First, notice that we can write an ℓ -cycle as a product of $\ell - 1$ transpositions. Therefore, even length cycles are odd permutations and odd length cycles are even permutations (confusing but true). Thus the 3-cycle (123) is an even permutation.

Next, notice that if σ can be written as a product of ℓ transpositions and τ can be written as a product of k transpositions, then $\sigma\tau$ can be written as a product of $\ell + k$ transpositions. Then we just recall that “even plus even is even” “odd plus odd is even” and “even plus odd is odd”. So two even or two odd permutations multiplied (i.e. composed) together give us an even permutation and an odd and an even permutation multiplied together give us an odd permutation.

Example: $(123)(45)(6789)$ is even since $(123) = \text{even}$, $(45) = \text{odd}$, and $(6789) = \text{odd}$, so even + odd + odd = even. Alternatively, $(123)(45)(6789) = (13)(12)(45)(69)(68)(67)$ — 6 transpositions, therefore, even.

THE SIGN HOMOMORPHISM:

Since we have well-defined notions of even and odd-ness, we can now define the map:

$$\text{sgn}(\sigma) = (-1)^\sigma = \begin{cases} +1 & \sigma \text{ is even} \\ -1 & \sigma \text{ is odd} \end{cases}$$

This map is called the “sign homomorphism”. It can be used to define the determinant of a matrix. Let $A = (a_{ij})$ be an $n \times n$ matrix with entries a_{ij} . Then

$$\det(A) = \sum_{\sigma \in S_n} (-1)^\sigma a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}$$

In particular, consider a 2×2 matrix. $S_2 = \{(1), (12)\}$. Let $\sigma = (1)$. σ is even so $(-1)^\sigma = +1$. Also, let $\tau = (12)$. τ is odd so $(-1)^\tau = -1$. Thus $\det(A) = (-1)^\sigma a_{1\sigma(1)} a_{2\sigma(2)} + (-1)^\tau a_{1\tau(1)} a_{2\tau(2)} = a_{11}a_{22} - a_{12}a_{21}$ (the regular determinant formula).

THE ALTERNATING GROUP:

From the last discussion we see that: even composed with even is even. Notice that the identity is always even and also the inverse of an even permutation is even (if $\sigma = (a_1 a_2) \cdots (a_{\ell-1} a_\ell)$, then $\sigma^{-1} = (a_{\ell-1} a_\ell) \cdots (a_1 a_2)$ — the same number of transpositions works for both σ and σ^{-1}). Putting this together we arrive at the following:

Theorem: For any $n > 1$, $A_n = \{\sigma \in S_n \mid \sigma \text{ is even}\}$ is a subgroup of S_n .

A_n is called the **alternating group** on n characters. These groups are very important. In fact, A_n is a non-abelian **simple** group when $n \geq 5$ (whatever that means).

Examples: $A_2 = \{(1)\}$ and $A_3 = \{(1), (123), (132)\}$.

Our previous example, $(123)(45)(6789)$, is an element of A_9 .

Notice that multiplying by (12) sends even permutations to odd permutations and vice-versa. The map $L(\sigma) = (12)\sigma$ is a bijection from A_n to the set of odd permutations. Thus exactly half of the permutations in S_n are even and half are odd. This implies that the order of A_n is $n!/2$. For example: $|A_3| = 3!/2 = 3$ and $|A_4| = 4!/2 = 12$.