Math 3110-101 **Homework #7** **Due: Wed., Apr. 29^{th}, 2015**

1. In each of the following rings, R: state the characteristic of the ring, give an example of a unit and its inverse (other than 1), and give an example of a zero divisor. If no unit exists, explain why not. If no zero divisors exist, explain why not.
   (a) $R = \mathbb{Z}_4 \oplus \mathbb{Z}_6$
   (b) $R = 3\mathbb{Z} = \{3k \mid k \in \mathbb{Z}\}$ (multiples of 3)
   (c) $R = (\mathbb{Z}_3)^{2\times 2}$ (2×2 matrices with entries in $\mathbb{Z}_3$)
2. Recall $R \oplus S$ is the direct product of the rings R and S.
   (a) Suppose R and S have 1's. Then show $R \oplus S$ is also a ring with 1.
   (b) Let R and S be rings with 1. Prove $U(R\oplus S) \cong U(R) \oplus U(S)$ (i.e. the group of units of $R\oplus S$ is isomorphic to the direct product of the group of units of R and the group of units of S). *Hint:* You will need to write down a (group) isomorphism.
3. Let $S = \left\{ \begin{bmatrix} x & 0 \\ x & 0 \end{bmatrix} \,\middle|\, x \in \mathbb{R} \right\}$.
   (a) Show S is a subring of $\mathbb{R}^{2\times 2}$ (the ring of 2×2 real matrices).
   (b) Is S commutative?
   (c) Does S have a unity (i.e. a multiplicative identity)? If so, what is it?
   (d) Is S an integral domain? A field?
4. Let R be a ring and let I and J be ideals in R. Note that $I + J = \{x + y \mid x \in I \text{ and } y \in j\}$, $I \cap J$, and $IJ = \{x_1y_1 + \cdots + x_ky_k \mid x_i \in I \text{ and } y_i \in J\}$ are ideals of R.
   (a) Prove that $I + J = \{x + y \mid x \in I \text{ and } y \in j\}$ is an ideal of R.
   (b) Consider the principle ideals (4) and (6) in $\mathbb{Z}$. What ideal do we get when we add them together: $(4)+(6)$? Intersect: $(4) \cap (6)$? Multiply $(4)(6)$?
   (c) Make a conjecture about the relationship between $m, n, d, \ell, p \in \mathbb{Z}$ if as ideals we have "$(m) + (n) = (d)$, $(m) \cap (n) = (\ell)$, and $(m)(n) = (p)$".
5. Let $I = (x^2)$. Create addition and multiplication tables for $\mathbb{Z}_2[x]/I$. Is this quotient ring a field?