Quotient Example

Example: Consider the quotient group ${\mathbb{Z}_6 \times \mathbb{Z}_{12}} \big/ {\langle 3 \rangle \times \langle 4 \rangle}$. For cleaner notation let $H = \langle 3 \rangle \times \langle 4 \rangle$.

Notice that since $\langle 3 \rangle$ is a cyclic **subgroup** of $\mathbb{Z}_6$ and $\langle 4 \rangle$ is a cyclic **subgroup** of $\mathbb{Z}_{12}$, we have that H is a subgroup of $\mathbb{Z}_6 \times \mathbb{Z}_{12}$. Also, since $\mathbb{Z}_6 \times \mathbb{Z}_{12}$ is Abelian, H is automatically a normal subgroup and thus quotienting by it actually makes sense.

Next, notice that $|\mathbb{Z}_6 \times \mathbb{Z}_{12}| = |\mathbb{Z}_6| \cdot |\mathbb{Z}_{12}| = 6 \cdot 12 = 72$. We have $|H| =$ (order of 3 in $\mathbb{Z}_6$)(order of 4 in $\mathbb{Z}_{12}$) $= 2 \cdot 3 = 6$. In particular,

$$H = \langle 3 \rangle \times \langle 4 \rangle = \{0, 3\} \times \{0, 4, 8\} = \{(0,0), (0,4), (0,8), (3,0), (3,4), (3,8)\}$$

where our first coordinates lie in $\mathbb{Z}_6$ and our second coordinates lie in $\mathbb{Z}_{12}$. Finally, by Lagrange's theorem, we have that our quotient group ${\mathbb{Z}_6 \times \mathbb{Z}_{12}} \big/ H$ has order $\frac{72}{6} = 12$.

Let's consider the element $(1,5) + H \in {\mathbb{Z}_6 \times \mathbb{Z}_{12}} \big/ H$. Its inverse is

$$-((1,5) + H) = -(1,5) + H = (-1,-5) + H = (5,7) + H$$

Note: Keep in mind that we use additive notation because $\mathbb{Z}_n$'s have additions as their group operation. We find an inverse of a coset by inverting the representative of our coset. We find the inverse of an element of a direct product by inverting each coordinate. Finally, we operate mod 6 in the first coordinate and mod 12 in the second.

Let's compute the order of $(1,5) + H$ as an element of the group ${\mathbb{Z}_6 \times \mathbb{Z}_{12}} \big/ H$. Thus we ask, how many times do we need to add $(1,5) + H$ to itself before we get to the identity of our quotient group (i.e., $0 + H = H$)?

- $(1,5) + H \neq H$ since $(1,5) \notin H$.
- $((1,5) + H) + ((1,5) + H) = ((1,5) + (1,5)) + H = (2,10) + H \neq H$ since $(2,10) \notin H$.
- $((1,5) + H) + ((1,5) + H) + ((1,5) + H) = (3,15) + H = (3,3) + H \neq H$ since $(3,3) \notin H$.
- $((1,5) + H) + ((1,5) + H) + ((1,5) + H) + ((1,5) + H) = (4,8) + H \neq H$ since $(4,8) \notin H$.
- $((1,5) + H) + ((1,5) + H) + ((1,5) + H) + ((1,5) + H) + ((1,5) + H) = (5,13) + H = (5,1) + H \neq H$ since $(5,1) \notin H$.
- $((1,5) + H) + ((1,5) + H) + ((1,5) + H) + ((1,5) + H) + ((1,5) + H) + ((1,5) + H) = (6,6) + H = (0,6) + H \neq H$ since $(0,6) \notin H$.

The above calculation shows that up to the sixth additive power of $(1,5) + H$ still does not get us back to the identity of our quotient group. However, our quotient group has order 12 and thus, by Lagrange's theorem, its elements must have orders dividing 12 (i.e., 1, 2, 3, 4, 6, or 12). The above calculation rules out orders 1, 2, 3, 4, and 6. Therefore, $(1,5) + H$ has order 12 in ${\mathbb{Z}_6 \times \mathbb{Z}_{12}} \big/ H$. In other words, it is a **generator** of that group! We just learned that

$${\mathbb{Z}_6 \times \mathbb{Z}_{12}} \big/ H = \Big\langle (1,5) + H \Big\rangle \cong \mathbb{Z}_{12}$$