Homework #10

Due: Fri., Apr. 22nd, 2022

Please remember when submitting any work via email or in person to...

PUT YOUR NAME ON YOUR WORK!

- #1 Let R be a commutative ring with 1 and let $a \in R$. Recall that $(a) = \{ra \mid r \in R\}$ is the principal ideal generated by a. In a commutative ring with 1, one says that $a, b \in R$ are **associates** if there exists some unit u (i.e., $u \in U(R)$) such that a = ub.
 - (a) What are the associates of -1 + 3i in $\mathbb{Z}[i]$?
 - (b) What are the associates of -1 + 3i in \mathbb{C} ?
 - (c) Let R be an integral domain. Show that (a) = (b) if and only if a and b are associates.
- #2 Let $I = \{a + bi \mid a \text{ and } b \text{ are even integers}\}$. Prove that $I \triangleleft \mathbb{Z}[i]$.
- #3 Let R be a ring and let $I, J \triangleleft R$. One defines $I + J = \{x + y \mid x \in I \text{ and } y \in J\}$ to be the sum of I and J. Similarly, one defines $IJ = \{x_1y_1 + \cdots + x_\ell y_\ell \mid \text{ for some } \ell \geq 0 \text{ and } x_1, \ldots, x_\ell \in I \text{ and } y_1, \ldots, y_\ell \in J\}$ to be the product of I and J. It can be shown that I + J, IJ, and $I \cap J$ are ideals of R.
 - (a) Prove that $I + J \triangleleft R$.
 - (b) Consider I = (12) and J = (10) in \mathbb{Z} . Calculate I + J, IJ, and $I \cap J$.
 - (c) Since \mathbb{Z} is a principal ideal domain, (a)(b) = (c), (a) + (b) = (d), and $(a) \cap (b) = (\ell)$ for some $c, d, \ell \in \mathbb{Z}$. Make a conjecture about how a and b (where a and b are non-zero) are related to c, d, and ℓ . Note: You don't have to prove your conjectures. Just tell me how you think this "ideal arithmetic" works.
- #4 As a quick reminder, in \mathbb{Z} and in \mathbb{Z}_n , we know that subgroup = normal subgroup = cyclic subgroup = subring = ideal = principal ideal.
 - (a) Let R be a finite commutative ring with 1. Explain why ideals are prime if and only if they are maximal.
 - (b) Find all the ideals of \mathbb{Z}_{24} and draw the corresponding lattice. Which ideals are prime? Which are maximal?
 - (c) Determine which ideals in \mathbb{Z} are prime and which are maximal. [Prove your assertions.] *Note:* Don't forget to consider the trivial ideal: $\{0\}$.
- RESUBMIT Type up Homework #9 Problem #4 and its solution in LATEX.

Prove that $\mathbb{Q}[\sqrt{5}] = \{a + b\sqrt{5} \mid a, b \in \mathbb{Q}\}\$ is a field.

Notes: Use a subring (of \mathbb{R}) test to show it is a ring. Use a "conjugate trick" to compute multiplicative inverses. Don't forget to mention the extra "trivial" bits about being a field: It is a *commutative* ring with unity. Specifically note $1 = 1 + 0\sqrt{5} \in \mathbb{Q}[\sqrt{5}]$ (and of course $1 \neq 0$).

When typing this problem up, please write carefully: Restate the problem. Write in complete sentences.