Math 3110 # Homework #1 **Due: Fri., Jan. 26th, 2024**

Please remember when submitting any work via email or in person to...

PUT YOUR NAME ON YOUR WORK!

#1. Group axiom basics.

(a) We know how to add, subtract, multiply, and divide real numbers: $\mathbb{R}$. In addition, we now know that $\mathbb{R}$ is a group under addition. Explain why each of these other operations fails to make $\mathbb{R}$ into a group.

Give concrete counterexamples.

Example: If I wanted to show $\mathbb{Z}$ is not a group under multiplication, I would say something like "The integers, $\mathbb{Z}$, do not form a group under multiplication because of a lack of multiplicative inverses. For example, $2 \in \mathbb{Z}$ but $2^{-1} = \frac{1}{2} \notin \mathbb{Z}$."

(b) Let's equip $\mathbb{Z}$ with a really weird operation: $x \circ y = x + y + 3$. For example, $-1 \circ 5 = -1 + 5 + 3 = 7$ and $(x \circ y) \circ z = (x + y + 3) \circ z = (x + y + 3) + z + 3 = x + y + z + 6$.

Obviously, $\mathbb{Z}$ is closed under $\circ$. Show $\mathbb{Z}$ is a group under this operation.

You need to show $\circ$ is associative, has an identity, and elements have inverses.

Also, is $\mathbb{Z}$ equipped with $\circ$ an abelian group?

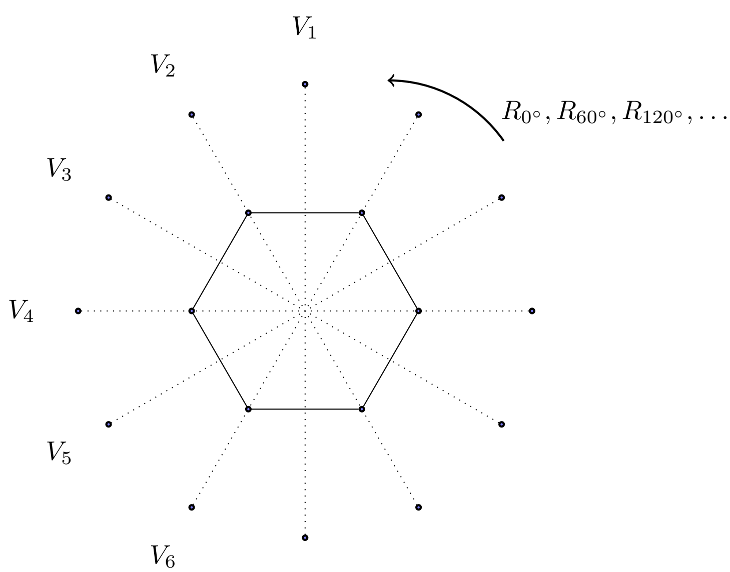
#2. Let G be a group with identity $e \in G$. In general, the law of exponents $(ab)^n = a^n b^n$ may fail to hold.

(a) Give a concrete example of a group G and elements $a, b \in G$ where $(ab)^2 \neq a^2 b^2$.

(b) Prove G is an abelian group if and only if for all $a, b \in G$, $(ab)^2 = a^2 b^2$.

Note: This is an "if and only if" statement. You need to prove two implications.

#3. Consider the dihedral group $D_6 = \{R_{0^\circ}, R_{60^\circ}, R_{120^\circ}, R_{180^\circ}, R_{240^\circ}, R_{300^\circ}, V_1, V_2, V_3, V_4, V_5, V_6\}$ (symmetries of a regular hexagon). [Rotations are done counter-clockwise and reflections are labeled in the picture below.]



(a) Compute $V_1 R_{60^\circ}$, $R_{180^\circ} V_4$, and $V_3 V_6$. [Draw some pictures!]

(b) Is D_6 Abelian? Why or why not?

(c) Make a table of inverses and orders for each element:

Element: $g =$	R_{0°	R_{60°	R_{120°	R_{180°	R_{240°	R_{300°	V_1	V_2	V_3	V_4	V_5	V_6
Inverse: $g^{-1} =$	???	$\cdots$										
Order: $\lvert g \rvert =$	???	$\cdots$										

Note: Recall that the order of an element g is the smallest positive power n such that g^n is the identity. If no such power exists the order of g is ∞.