Math 3110 # Homework #7 **Due: Mon., Mar. 25th, 2024**

Please remember when submitting any work via email or in person to...

PUT YOUR NAME ON YOUR WORK!

#1 Suppose H and K are subgroups of some group G such that $|H| = 66$ and $|K| = 42$. Just considering divisibility criteria coming from Lagrange's Theorem, what is the largest size the subgroup $H \cap K$ could be? Suppose M contains both H and K. What is the smallest size M could be?

#2 Direct products.[1]

(a) Find the order of $(6, 2, -k)$ in $\mathbb{Z}_{22} \times U(9) \times Q$ where $Q = \{\pm 1, \pm i, \pm j, \pm k\}$ is the quaternion group.

(b) Explain why $\mathbb{Z}_{20} \cong \mathbb{Z}_4 \times \mathbb{Z}_5$ but $\mathbb{Z}_{20} \not\cong \mathbb{Z}_2 \times \mathbb{Z}_{10}$.
In addition, list the distinct orders of elements in both $\mathbb{Z}_4 \times \mathbb{Z}_5$ and $\mathbb{Z}_2 \times \mathbb{Z}_{10}$. Give an example of an element of each such order.

#3 Let H and K be normal subgroups of G. Show that $HK = \{hk \mid h \in H \text{ and } k \in K\}$ is a normal subgroup of G.

You need to run through the normal subgroup test.

Note: It may be helpful to note that $HK = KH$. Why? Given $x \in HK$, we have $x = hk$ for some $h \in H$ and $k \in K$. Therefore, $x = hk \in Hk$. But H is a normal subgroup of G so that $Hk = kH$. Therefore, $x \in kH$ so that there exists some $h' \in H$ such that $x = kh'$. In other words, $x \in KH$. This shows that $HK \subseteq KH$. The proof of the other containment is similar. For a slicker proof: $HK = \bigcup_{k \in K} Hk = \bigcup_{k \in K} kH = KH$ (where the middle step used that H is normal and thus its left and right cosets match).

#4 Let $H = \{1, x^3, y, x^3y\} \subseteq D_6 = \{1, x, \ldots, x^5, y, xy, \ldots, x^5y\} = \langle x, y \mid x^6 = 1, y^2 = 1, xyxy = 1\rangle$. Make a table showing H is closed under D_6's operation (thus is a subgroup by the finite subgroup test).

What is $[D_6 : H]$ (i.e. the index of H in D_6)?

Find all of the left and right cosets of H in D_6. Is H a normal subgroup of D_6?

#5 Let G and H be groups.

(a) Show $\{e\} \times H = \{(e, h) \mid h \in H\}$ is a normal subgroup of $G \times H$ (where e is the identity of G).
Note: You need to show that $\{e\} \times H$ is a subgroup AND that it's normal.

(b) Show $G \times H \cong H \times G$.

[1] Gallian uses $G \oplus H$ for direct products. I will use the more standard $G \times H$ notation.