Math 3110 # Homework #8 **Due: ~~Mon., Apr. 1st, 2024~~ Wed., Apr. 3rd, 2024**

Please remember when submitting any work via email or in person to...

PUT YOUR NAME ON YOUR WORK!

#1 Let $H = \langle x^2 \rangle = \{1, x^2, x^4, x^6\} \subseteq D_8 = \langle x, y \mid x^8 = 1, y^2 = 1, (xy)^2 = 1 \rangle = \{1, x, \ldots, x^7, y, xy, \ldots, x^7 y\}$.

(a) Compute the (distinct) left and right cosets of H to confirm this is a normal subgroup of D_8.

(b) Write down a Cayley table for D_8/H.

(c) Is D_8/H abelian? Is it cyclic? Explain you answers.

#2 Quotients in $\mathbb{Z}_n$.

(a) Let $H = \langle 18 \rangle \subseteq \mathbb{Z}_{30}$.

i. List the elements of H (i.e., $H = \{???\}$) and explain why it is a normal subgroup of $\mathbb{Z}_{30}$.

ii. Compute the (distinct) cosets of H in $\mathbb{Z}_{30}$.

iii. Write down a Cayley table for $\mathbb{Z}_{30}/H$. What familiar group is this isomorphic to?

(b) Consider $k, \ell, n \in \mathbb{Z}_{>0}$ where $n = k\ell$. We attempt to define a function $\varphi : \mathbb{Z}_n \to \mathbb{Z}_\ell$ by $\varphi(x) = x$.
[Using more precise notation: $\varphi(x + n\mathbb{Z}) = x + \ell\mathbb{Z}$.]

Prove that φ is a *well-defined* homomorphism.

Determine φ's kernel and image. Then apply the first isomorphism theorem.

#3 Recall $\mathrm{GL}_n(\mathbb{R})$ (equipped with matrix multiplication) is the group of $n \times n$ invertible matrices. Also, recall $\mathbb{R}_{\neq 0}$ is a group under multiplication. Explain why the determinant map, $\det : \mathrm{GL}_n(\mathbb{R}) \to \mathbb{R}_{\neq 0}$, is a homomorphism. Show det is onto. What is its kernel? What does the first isomorphism theorem say here?

#4 Let $H \triangleleft G$ where G is a finite group. Suppose $xH \in G/H$ and xH is an element of order m in G/H. Show that G must have an element of order m.

Warning: Just because xH has order m (in G/H) does not mean x has order m in G.
So x itself might not work, but the right *power* of x will.

#5 RESUBMIT Type up Homework #7 Problem #5(a) and its solution in LaTeX.

> Let G and H be groups.
>
> Show $\{e\} \times H = \{(e, h) \mid h \in H\}$ is a normal subgroup of $G \times H$ (where e is the identity of G).
>
> *Note:* You need to show that $\{e\} \times H$ is a subgroup AND that it's normal in $G \times H$.

When typing this problem up, please write carefully: Restate the problem. Write in complete sentences.