Math 3110-101 # Homework #9 **Due: Fri., Apr. 19th, 2024**

Please remember when submitting any work via email or in person to...

PUT YOUR NAME ON YOUR WORK!

#1 In each of the following rings, R, state the characteristic of the ring. If R has unity, give an example of a unit and its inverse (other than 1 itself). If no unit exists, explain why not. If R has zero divisors, give an example of a zero divisor. If no zero divisors exist, explain why not.

(a) $R = \mathbb{Z}_{14} \times \mathbb{Z}_{10}$

(b) $R = (2\mathbb{Z})[x]$ (polynomials with even integer coefficients)

(c) $R = (\mathbb{Z}_5)^{2\times 2}$ (2×2 matrices with entries in $\mathbb{Z}_5$)

#2 Let R be a ring with 1. Show that $r \in R$ cannot be both a zero divisor and a unit.

If $u, v \in R$ are units, is uv always a unit? [Explain why or give a concrete counterexample.]

If $a, b \in R$ are zero divisors, is ab always a zero divisor? [Explain why or give a concrete counterexample.]

Note: Be careful. Do not assume R is commutative.

Also, pay attention to the fact that zero divisors are by definition *nonzero* elements.

#3 Let $S = \left\{ \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \,\middle|\, a, b \in \mathbb{Z} \right\} \subseteq \mathbb{Z}^{2\times 2}$.

(a) Prove S is a subring of $\mathbb{Z}^{2\times 2}$.

(b) Show S is a commutative ring with 1.

(c) Recall that $U(S)$ are the units of S. Find all of them. What kind of group is the group of units of S?

Note: $U(\mathbb{Z}^{2\times 2}) = \mathrm{GL}_2(\mathbb{Z}) = \{A \in \mathbb{Z}^{2\times 2} \mid \det(A) = \pm 1\}$.

The determinant should be helpful in figuring out what the units of S are.

(d) $\mathbb{Z}^{2\times 2}$ has zero divisors but S does not. Give an example of a zero divisor in $\mathbb{Z}^{2\times 2}$.

Next, we can show S has no zero divisors. Fill in the details:
Suppose A is a zero divisor in S ($\subseteq \mathbb{Z}^{2\times 2} \subseteq \mathbb{R}^{2\times 2}$). Explain why A^{-1} cannot exist in $\mathbb{R}^{2\times 2}$.
What does this say about $\det(A)$? What matrices in S have such a determinant?

Note: Since S is a commutative ring with $1 \neq 0$ and has no zero divisors, it is called an *integral domain*.

#4 Prove that $\mathbb{Q}[\sqrt{-3}] = \{a + b\sqrt{-3} \mid a, b \in \mathbb{Q}\}$ is a field.

Note: Use a subring test to show it is a ring (it is a subring of $\mathbb{C}$).

Then use the "conjugate trick" to show it has multiplicative inverses.