Math 3110 **Homework #8** **Due: Mon., Mar. 31st, 2025**

Please remember when submitting any work via email or in person to...

PUT YOUR NAME ON YOUR WORK!

#1 Recall that the center of a group is a normal subgroup. We are given that $Z = Z(D_6) = \{1, x^3\}$ where $D_6 = \langle x, y \mid x^6 = 1, y^2 = 1, (xy)^2 = 1\rangle = \{1, x, \ldots, x^5, y, xy, \ldots, x^5y\}$.

Find the distinct cosets of Z in D_6 then write down a Cayley table for D_6/Z.

Is D_6/Z abelian? Is it cyclic? Explain your answer.

#2 Quotients in $\mathbb{Z}_n$.

(a) Let $H = \langle 20\rangle \subseteq \mathbb{Z}_{50}$. List the elements of H (i.e., $H = \{???\}$). Then compute the cosets of H in $\mathbb{Z}_{50}$. What familiar group is $\mathbb{Z}_{50}/H$ isomorphic to?

(b) Let k, ℓ, n be positive integers such that $n = k\ell$ (i.e. ℓ divides n). It turns out that $\mathbb{Z}_n/\langle \ell\rangle \cong \mathbb{Z}_\ell$.
Prove this using the map $\varphi : \mathbb{Z}_n \to \mathbb{Z}_\ell$ defined by $\varphi(x) = x$ and the first isomorphism theorem.
Note: You need to show φ is a **well-defined** homomorphism, then find φ's kernel and image, and finally, apply the first isomorphism theorem.

#3 Let $\varphi : D_n \to \{\pm 1\}$ be defined by $\varphi(a) = \begin{cases} +1 & a \text{ is a rotation} \\ -1 & a \text{ is a reflection} \end{cases}$

Show φ is a homomorphism. What is the kernel of φ? What does the first isomorphism theorem tell us here?

#4 Let G be a finite group, H a normal subgroup of G, and $g \in G$. Show that $|gH|$ divides $|g|$ (in G).

Note: In this problem, $|gH|$ means the order of the element gH in the quotient group G/H (as opposed to the cardinality of gH as a set).

Unnecessary Note: This holds for all groups if one uses the convention: all positive integers as well as infinity itself divides infinity.

#5 Let G and H be finite groups and let $\varphi : G \to H$ be an epimorphism (this is a homomorphism which is onto).

Suppose that there is some $x \in H$ such that $|x| = 8$. Prove that G has an element of order 8 as well.

RESUBMIT Type up Homework #7 Problem #4(a) and its solution in LaTeX.

> Let G and H be groups. Show $\{e\} \times H = \{(e, h) \mid h \in H\}$ is a normal subgroup of $G \times H$ (where e is the identity of G).
>
> *Note:* You need to show that $\{e\} \times H$ is a subgroup AND that it's normal in $G \times H$.

When typing this problem up, please write carefully: Restate the problem. Write in complete sentences.