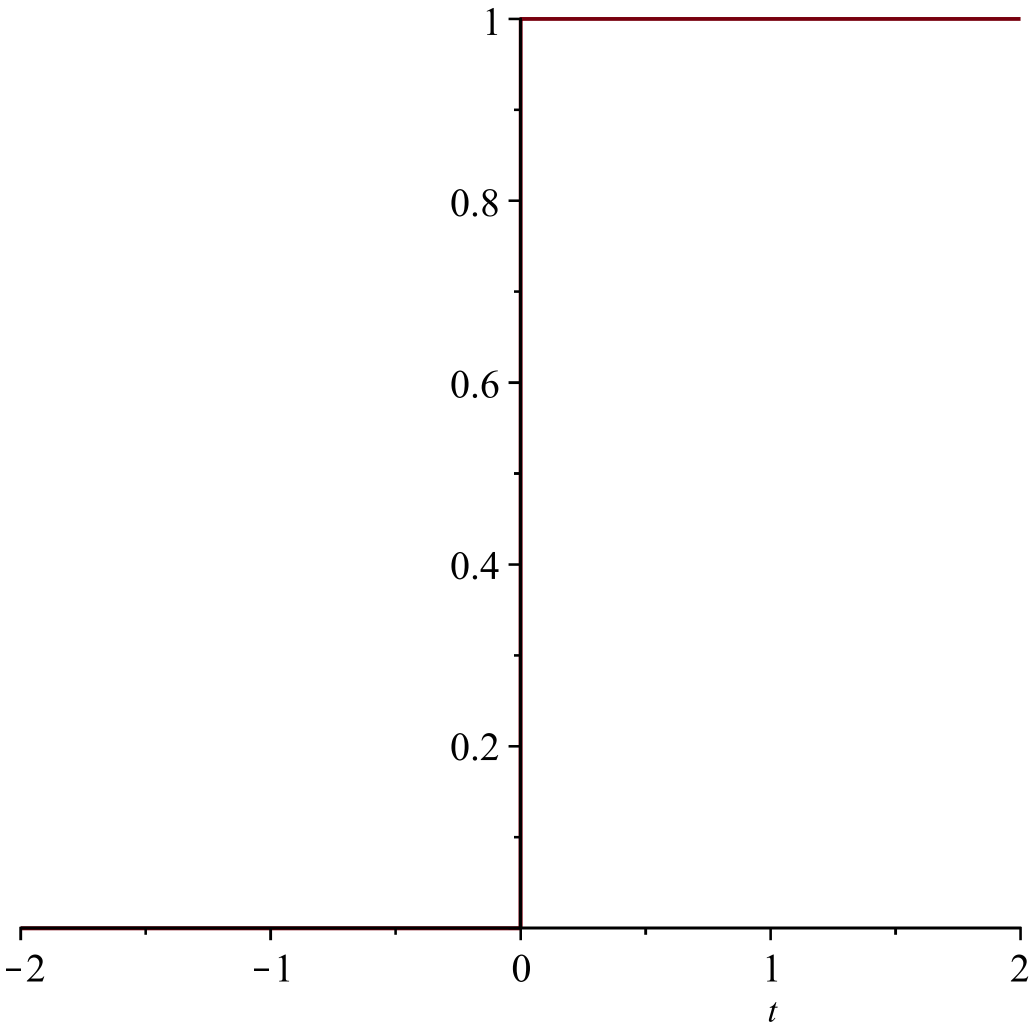
Math 3130 - Laplace Transform Examples

```
> restart;
  with(inttrans):
> plot(Heaviside(t),t=-2..2);
```



Let's solve: $y' = -3\,y + u_1(t) - u_2(t)$ with y(0) = 2.

Note: In Maple, $u_a(t) = \text{Heaviside}(t-a)$. Maple's only Heaviside function is the one based at zero, so we must shift it.

```
> laplace(diff(y(t),t)=-3*y(t)+Heaviside(t-1)-Heaviside(t-2),t,s);
```

$$s\,laplace(y(t),t,s) - y(0) = \frac{e^{-s} - e^{-2s}}{s} - 3\,laplace(y(t),t,s) \quad (1)$$

```
> Y = solve(s*laplace(y(t), t, s) - y(0) = (exp(-s) - exp(-2*s))/s - 3*
  laplace(y(t), t, s),laplace(y(t),t,s));
```

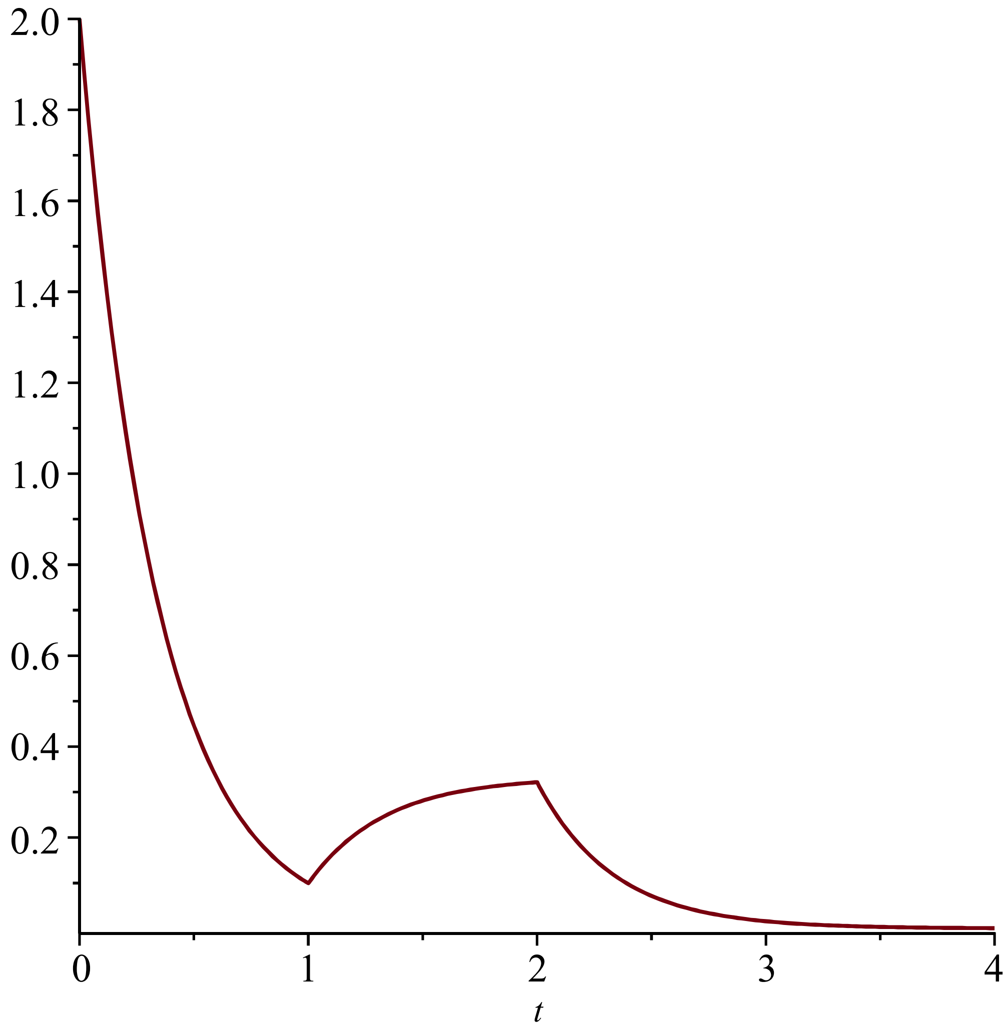
$$Y = \frac{y(0)\,s + e^{-s} - e^{-2s}}{s\,(3+s)} \quad (2)$$

```
> invlaplace(laplace(y(t), t, s)=(2*s + exp(-s) - exp(-2*s))/(s*(3 + s)
  ),s,t);
```

$$y(t) = 2\,e^{-3t} + \frac{\text{Heaviside}(t-1)\,\left(1 - e^{-3t+3}\right)}{3} - \frac{\text{Heaviside}(t-2)\,\left(1 - e^{-3t+6}\right)}{3} \quad \textbf{(3)}$$

Thus $y(t) = 2e^{-3t} + \frac{1}{3}u_1(t)\left(1 - e^{-3(t-1)}\right) - \frac{1}{3}u_2(t)\left(1 - e^{-3(t-2)}\right)$

```
> plot(2*exp(-3*t) + Heaviside(t - 1)*(1 - exp(-3*t + 3))/3 - Heaviside
  (t - 2)*(1 - exp(-3*t + 6))/3,t=0..4);
```



Partial Fractions Example:

```
> convert(2*s/((s^2+1)*(s^2+9)), parfrac, s);
```

$$-\frac{s}{4\left(s^2+9\right)} + \frac{s}{4\left(s^2+1\right)} \quad \textbf{(4)}$$

Solve y'' + 9y = 2sin(3t) with y(0) = y'(0) = 0.

[We have resonance.]

```
> laplace(diff(y(t),t,t)+9*y(t)=2*sin(3*t),t,s);
```

$$s^2\,\mathit{laplace}(y(t), t, s) - D(y)(0) - y(0)\,s + 9\,\mathit{laplace}(y(t), t, s) = \frac{6}{s^2+9} \quad \textbf{(5)}$$

```
> Y = solve(s^2*laplace(y(t), t, s) + 9*laplace(y(t), t, s) = 6/(s^2 +
  9),laplace(y(t),t,s));
```

$$Y = \frac{6}{\left(s^2+9\right)^2} \tag{6}$$

```
> invlaplace(6/(s^2+9)^2,s,t);
```

$$\frac{\sin(3\,t)}{9} - \frac{t\cos(3\,t)}{3} \tag{7}$$

Solution: $y(t) = \frac{1}{9}\sin(3\,t) - \frac{1}{3}t\cdot\cos(3\,t)$.

```
> Dirac(t);
```

$$\mathrm{Dirac}(t) \tag{8}$$

Solve: y'' + 2y' + 2y = $\delta_{20}(t)$ with y(0) = 5 and y'(0) = 0.

[We have an underdamped harmonic oscillator starting 5 units from the equilibrium. Then 20 time units in it is struck with an impulse force.]

```
> laplace(diff(y(t),t,t)+2*diff(y(t),t)+2*y(t)=Dirac(t-20),t,s);
```

$$s^2\,laplace(y(t),t,s) - \mathrm{D}(y)(0) - y(0)\,s + 2\,s\,laplace(y(t),t,s) - 2\,y(0) + 2\,laplace(y(t),t,s) = \mathrm{e}^{-20\,s} \tag{9}$$

```
> Y = solve(s^2*laplace(y(t), t, s) - 0 - 5*s + 2*s*laplace(y(t), t, s)
  - 2*5 + 2*laplace(y(t), t, s) = exp(-20*s), laplace(y(t), t, s));
```

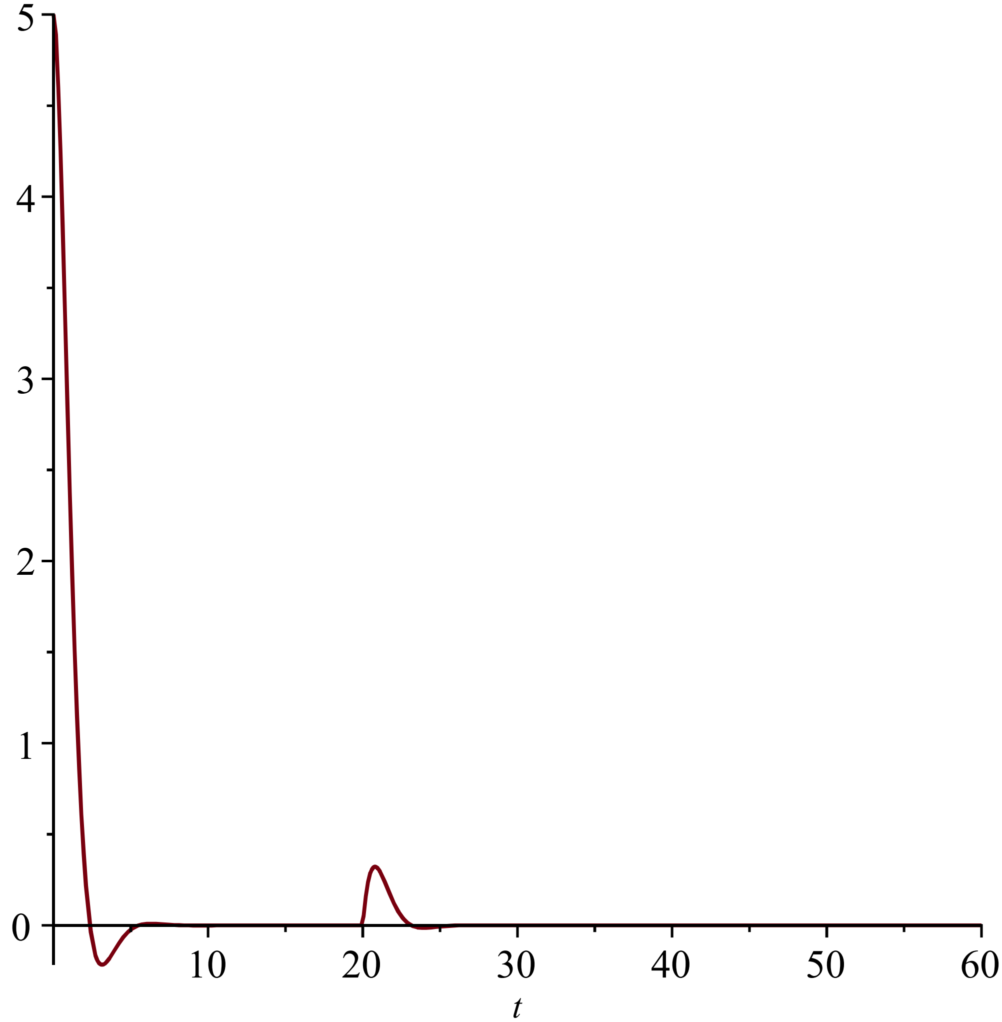
$$Y = \frac{\mathrm{e}^{-20\,s} + 5\,s + 10}{s^2 + 2\,s + 2} \tag{10}$$

```
> invlaplace((exp(-20*s) + 5*s + 10)/(s^2 + 2*s + 2),s,t);
```

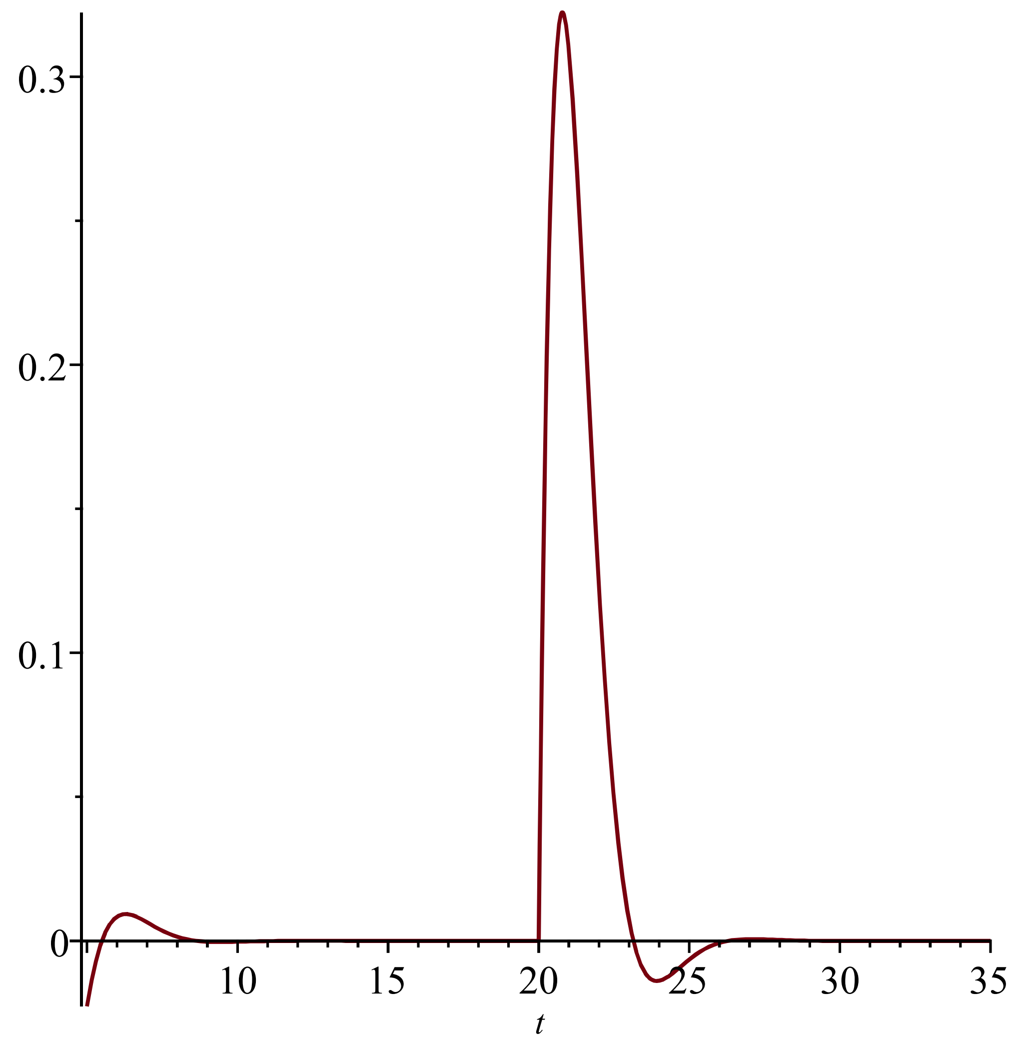
$$\mathrm{Heaviside}(t-20)\,\sin(t-20)\,\mathrm{e}^{-t+20} + 5\,\mathrm{e}^{-t}\,(\cos(t) + \sin(t)) \tag{11}$$

Solution: y(t) = $5\,\mathrm{e}^{-t}\cos(t) + 5\,\mathrm{e}^{-t}\sin(t) + u_{20}(t)\,\mathrm{e}^{-(t-20)}\sin(t-20)$.

```
> plot(Heaviside(t - 20)*exp(-t + 20)*sin(t - 20) + 5*exp(-t)*(cos(t) +
  sin(t)),t=0..60);
```




```
> plot(Heaviside(t - 20)*exp(-t + 20)*sin(t - 20) + 5*exp(-t)*(cos(t) +
  sin(t)),t=5..35);
```


0.3
0.2
0.1
0
10
15
20
25
30
35
t