Math 3130 | **Homework #3** | **Due: June 19^{th}, 2019**

Be sure to show your work!

You ~~may~~ **should** use technology to complete this assignment. To avoid miscopying errors, I have provided definitions for several coefficient matrices that can be copied/pasted into Maple. Recall that to use linear algebra related commands you need to execute `with(LinearAlgebra):` Also, remember that the command: `Eigenvectors(A);` computes the eigenvalues and eigenvectors of a matrix A.

1. Solving homogeneous linear systems. In each case, first find the general solution and then solve the initial value problem.

(a) $\mathbf{y}' = A\mathbf{y}$ where $A = \begin{bmatrix} 11 & 24 & -9 \\ -6 & -13 & 6 \\ -6 & -12 & 8 \end{bmatrix}$ and given the initial value: $\mathbf{y}(0) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

Maple input: `A := <<11|24|-9>,<-6|-13|6>,<-6|-12|8>>;`

(b) $\mathbf{y}' = A\mathbf{y}$ where $A = \begin{bmatrix} 5 & 6 & -6 \\ -6 & -7 & 6 \\ -3 & -3 & 2 \end{bmatrix}$ and given the initial value: $\mathbf{y}(0) = \begin{bmatrix} 4 \\ 2 \\ -1 \end{bmatrix}$.

Maple input: `A := <<5|6|-6>,<-6|-7|6>,<-3|-3|2>>;`

For the next part (1(c)), we need a different approach. In 1(c), you cannot build a general solution from eigenvectors since there aren't enough of them! Instead we use the magic bullet that is the matrix exponential.

Note that $e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!} = I + A + \frac{A^2}{2} + \cdots$ is a convergent series (of matrices) for any matrix A. It isn't too hard to show that $\frac{d}{dt}\left[e^{At}\right] = Ae^{At}$ (as we might guess). This means that if $\mathbf{y}(t) = e^{At}\mathbf{c}$, then $\mathbf{y}'(t) = Ae^{At}\mathbf{c} = A\mathbf{y}(t)$. We have that $\mathbf{y}(t) = e^{At}\mathbf{c}$ is the general solution of $\mathbf{y}' = A\mathbf{y}$. More than that $\mathbf{y}(0) = e^{A(0)}\mathbf{c} = e^0\mathbf{c} = I\mathbf{c} = \mathbf{c}$. So if our initial condition is $\mathbf{y}(0) = \mathbf{y}_0$, then the IVP solution is $\mathbf{y}(t) = e^{At}\mathbf{y}_0$.

Computing the matrix exponential is quite involved. Fortunately, Maple can do this for us with the LinearAlgebra package's `MatrixExponential` command.

(c) $\mathbf{y}' = A\mathbf{y}$ where $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 1 & -1 \\ 5 & 8 & 6 \end{bmatrix}$ and given the initial value: $\mathbf{y}(0) = \begin{bmatrix} 4 \\ 2 \\ -1 \end{bmatrix}$.

Maple input: `A := <<2|-1|0>,<-1|1|-1>,<5|8|6>>;`

2. Classify the equilibrium solution of $\mathbf{y}' = A\mathbf{y}$ for the following matrices (i.e., is it a node sink, spiral source, ...?). State whether it is stable, asymptotically stable, or unstable. Then create a phase portrait (use any software you want or do it by hand, but whatever you use, make it accurate).

(a) $\begin{bmatrix} 2 & -5 \\ -5 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 2 \\ -3 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} -4 & 1 \\ 1 & -4 \end{bmatrix}$ (d) $\begin{bmatrix} -1 & 4 \\ -2 & 1 \end{bmatrix}$

3. Consider $A = \begin{bmatrix} -2 & 2 \\ 4 & -4 \end{bmatrix}$.

(a) Find eigenvalues and corresponding eigenvectors for A.
(b) Write down the general solution of $\mathbf{y}' = A\mathbf{y}$.
(c) What are the equilibrium solutions?
(d) Create a phase portrait.

4. Consider $A = \begin{bmatrix} \alpha & \alpha \\ 1 & 0 \end{bmatrix}$. We get a one-parameter system: $\mathbf{y}' = A\mathbf{y}$.

(a) Sketch the curve determined by α in the trace-determinant plane.
(b) Find the bifurcation values and describe what kind of equilibria we have before and after each bifurcation value.
(c) Pick sample values of α before, at, and after each bifurcation value and provide a phase portrait at those values.