# Darboux Integration

Jean Gaston Darboux provides us with a useful characterization of Riemann integrability. Instead of stating an integrability condition based on a generalized kind of limit, Darboux's condition is stated in terms of supremums and infimums.

**Notation:** Let $a, b \in \mathbb{R}$ with $a < b$. Suppose that $f : [a, b] \to \mathbb{R}$ is a **bounded** function. Given a partition $\mathcal{P} = \{x_0, \ldots, x_n\}$ of $[a, b]$ (i.e., $a = x_0 < x_1 < \cdots < x_n = b$), let $\Delta x_i = x_i - x_{i-1}$ (i.e., the width of the $i^{\text{th}}$-subinterval). $\|P\| = \max\{\Delta x_1, \ldots, \Delta x_n\}$ is the norm (or mesh size) of the partition. A sampling for $\mathcal{P}$ is a set $\mathcal{S} = \{s_1, \ldots, s_n\}$ where for each $i$, $s_i \in [x_{i-1}, x_i]$. Finally, $\text{RS}(f, \mathcal{P}, \mathcal{S}) = \sum_{i=1}^{n} f(s_i)\Delta x_i$ is the Riemann sum of $f$ relative to the partition $\mathcal{P}$ and sampling $\mathcal{S}$.

**Definition:** Let $\mathcal{P} = \{x_0, \ldots, x_n\}$ be a partition with $m_i = \inf_{x \in [x_{i-1}, x_i]} f(x)$ and $M_i = \sup_{x \in [x_{i-1}, x_i]} f(x)$ (these numbers exist because $f$ is bounded). We define...

$$\text{Upper Darboux Sum:} \quad U(f, \mathcal{P}) = \sum_{i=1}^{n} M_i \Delta x_i \qquad\qquad \text{Lower Darboux Sum:} \quad L(f, \mathcal{P}) = \sum_{i=1}^{n} m_i \Delta x_i$$

Let $\mathcal{S}$ be some sampling relative to $\mathcal{P}$. Then $m_i \leq f(s_i) \leq M_i$ (by definition of infimum and supremum), so $L(f, \mathcal{P}) \leq \text{RS}(f, \mathcal{P}, \mathcal{S}) \leq U(f, \mathcal{P})$.

**Lemma:** Given a partition $\mathcal{P}$, for every $\epsilon > 0$ there exists a sampling $\mathcal{S}_1$ such that $U(f, \mathcal{P}) - \text{RS}(f, \mathcal{P}, \mathcal{S}_1) < \epsilon$ and a sampling $\mathcal{S}_2$ such that $\text{RS}(f, \mathcal{P}, \mathcal{S}_2) - L(f, \mathcal{P}) < \epsilon$.
**proof:** Consider $\epsilon/(n\Delta x_i) > 0$. There exists some $s_i \in [x_{i-1}, x_i]$ such that $M_i - f(s_i) < \epsilon/(n\Delta x_i)$ because $M_i = \sup_{x \in [x_{i-1}, x_i]} f(x)$. This gives us our sampling $\mathcal{S}_1 = \{s_1, \ldots, s_n\}$.

$$U(f, \mathcal{P}) - \text{RS}(f, \mathcal{P}, \mathcal{S}_1) = \sum_{i=1}^{n} M_i \Delta x_i - \sum_{i=1}^{n} f(s_i)\Delta x_i = \sum_{i=1}^{n} (M_i - f(s_i))\Delta x_i < \sum_{i=1}^{n} \frac{\epsilon}{n\Delta x_i}\Delta x_i = \sum_{i=1}^{n} \frac{\epsilon}{n} = \epsilon$$

The proof for the lower sum is similar. ■

**Definition:**

$$\text{Upper Darboux Integral:} \quad \overline{\int_a^b} f = U(f) = \inf\{U(f, \mathcal{P} \mid \mathcal{P} \text{ is a partition}\}$$

$$\text{Lower Darboux Integral:} \quad \underline{\int_a^b} f = L(f) = \sup\{L(f, \mathcal{P} \mid \mathcal{P} \text{ is a partition}\}$$

Notice that the set of upper sums is bounded below by lower sums and vice-versa. Therefore, these upper and lower integrals *always exist*. We can think of $U(f, \mathcal{P})$ as an overestimate of the integral of $f$ and $L(f, \mathcal{P})$ as an underestimate. So more-or-less, this makes $U(f)$ the lowest of all of the overestimates and $L(f)$ the highest of all the underestimates.

**Definition:** Adding more points to a partition yields a **refinement**. In particular, if $\mathcal{P}$ and $\mathcal{P}'$ are partitions such that $\mathcal{P} \subseteq \mathcal{P}'$, we say that $\mathcal{P}'$ is a *refinement* of $\mathcal{P}$.

Notice that if $\mathcal{P}_1$ and $\mathcal{P}_2$ are partitions, then $\mathcal{P}_1 \cup \mathcal{P}_2$ is a *common refinement* of both $\mathcal{P}_1$ and $\mathcal{P}_2$. Also, it should be obvious that if $M = \sup_{x \in [z_0, z_m]} f(x)$ and $M_i = \sup_{x \in [z_{i-1}, z_i]} f(x)$, then $M = \max\{M_1, \ldots, M_n\}$. This means that $M(z_n - z_0) = \sum_{i=1}^{n} M\Delta z_i \geq \sum_{i=1}^{n} M_i \Delta z_i$. A similar statement follows for infimums. Thus we have that...

**Lemma:** Given a refinement of partitions $\mathcal{P} \subseteq \mathcal{P}'$, we have...

$$L(f, \mathcal{P}) \leq L(f, \mathcal{P}') \leq U(f, \mathcal{P}') \leq U(f, \mathcal{P})$$

In particular, given any two partitions, $\mathcal{P}_1$ and $\mathcal{P}_2$, we have...

$$L(f, \mathcal{P}_1) \leq L(f, \mathcal{P}_1 \cup \mathcal{P}_2) \leq U(f, \mathcal{P}_1 \cup \mathcal{P}_2) \leq U(f, \mathcal{P}_2)$$

Therefore, every lower sum is a lower bound for all upper sums and every upper sum is an upper bound for all lower sums. This then implies that $L(f, \mathcal{P}_1) \leq U(f)$ and $L(f) \leq U(f, \mathcal{P}_2)$ (by the definitions of infimum and supremum). Thus $U(f)$ is an upper bound for all lower sums and $L(f)$ is a lower bound for all upper sums. Thus because $L(f)$ and $U(f)$ are greatest lower and least upper bounds, so for all partitions $\mathcal{P}_1$ and $\mathcal{P}_2$...

$$L(f, \mathcal{P}_1) \leq L(f) \leq U(f) \leq U(f, \mathcal{P}_2)$$

Next, we a technical lemma. Its proof doesn't use anything fancy, it's just takes some effort to write down.

**Lemma:** Given a partition $\mathcal{P}'$ and $\epsilon > 0$, there is some $\delta > 0$ such that for every partition $\mathcal{P}$ with $\|\mathcal{P}\| < \delta$ we have $U(f,\mathcal{P}) - U(f,\mathcal{P}\cup\mathcal{P}') < \epsilon$. Similarly there is some $\delta > 0$ such that for every partition $\mathcal{P}$ with $\|\mathcal{P}\| < \delta$ we have $L(f,\mathcal{P}\cup\mathcal{P}') - L(f,\mathcal{P}) < \epsilon$. Putting these facts together (pick the minimum of two $\delta$'s), there is some $\delta > 0$ such that for every partition $\mathcal{P}$ with $\|\mathcal{P}\| < \delta$ we have both $U(f,\mathcal{P}) - U(f,\mathcal{P}\cup\mathcal{P}') < \epsilon$ and $L(f,\mathcal{P}\cup\mathcal{P}') - L(f,\mathcal{P}) < \epsilon$.

**proof:** Let $M \in \mathbb{R}$ such that $|f(x)| < M$ for all $x \in [a,b]$ ($M$ exists because we assumed that $f$ is bounded on $[a,b]$). Let $\mathcal{P}' = \{x_0,\dots,x_n\}$. Suppose $\epsilon > 0$ and let $\delta = \min\{\epsilon/(2nM), \Delta x_1, \dots, \Delta x_n\}$.

Consider any partition $\mathcal{P} = \{y_0,\dots,y_m\}$ such that $\|\mathcal{P}\| < \delta$. We want to analyze $\mathcal{P}\cup\mathcal{P}' = \{z_0,\dots,z_\ell\}$. Now $\|\mathcal{P}\| < \delta \le \min\{\Delta x_1,\dots,\Delta x_n\}$ says that the largest gap between two $y_j$'s is smaller than the smallest gap between two $x_k$'s. This means that in $\mathcal{P}\cup\mathcal{P}'$ every $x_k \in \mathcal{P}'$ (other than $x_0$ and $x_n$) is surrounded by $y_j$'s from $\mathcal{P}$. In particular, our partition looks like:

$$a = x_0 = y_0 < y_1 < \cdots < y_{i_1} \le x_1 < y_{i_1+1} < \dots y_{i_2} \le x_2 < y_{i_2+1} < \cdots < y_m = x_n = b$$

Let's consider each subinterval $[z_{j-1}, z_j]$.

*Case 1:* Suppose we move between $y_k$'s. That is, suppose $[z_{j-1},z_j] = [y_{k-1},y_k]$ for some $k$. Then...

$$\sup_{x\in[z_{j-1},z_j]} f(x)\Delta z_j = \sup_{x\in[y_{k-1},y_k]} f(x)\Delta y_k$$

So the corresponding terms in the summations $U(f,\mathcal{P})$ and $U(f,\mathcal{P}\cup\mathcal{P}')$ cancel out.

*Case 2:* Suppose we jump over an $x_k$. That is, $z_{j-1} = y_{i_k} < x_k = z_j < y_{i_k+1} = z_{j+1}$.

Let $M''' = \sup\limits_{x\in[y_{i_k},y_{i_k+1}]} f(x) = \sup\limits_{x\in[z_{j-1},z_{j+1}]} f(x)$, $M' = \sup\limits_{x\in[y_{i_k},x_k]} f(x) = \sup\limits_{x\in[z_{j-1},z_j]} f(x)$, and $M'' = \sup\limits_{x\in[x_k,y_{i_k+1}]} f(x) = \sup\limits_{x\in[z_j,z_{j+1}]} f(x)$. Then the $(i_k+1)$-st term in $U(f,\mathcal{P})$ is $M'''\Delta y_{i_k+1} = M'''(\Delta z_j + \Delta z_{j+1}) = M'''\Delta z_j + M'''\Delta z_{j+1}$ and the $j$-th and $(j+1)$-st terms in $U(f,\mathcal{P}\cup\mathcal{P}')$ are $M'\Delta z_j$ and $M''\Delta z_{j+1}$. Now since $M'$ and $M''$ are bounded above by $M'''$ and the function itself is bounded by $M$, we have $M'''-M' < M$ and $M'''-M'' < M$. Thus the difference between the $(i_k+1)$-st term in $U(f,\mathcal{P})$ and the $j$-th and $(j+1)$-st terms in $U(f,\mathcal{P}\cup\mathcal{P}')$ is $(M'''-M')\Delta z_j + (M'''-M'')\Delta z_{j+1} < M\Delta z_j + M\Delta z_{j+1} < 2M\|\mathcal{P}\cup\mathcal{P}'\| \le 2M\|\mathcal{P}\| < 2M\delta \le 2M\dfrac{\epsilon}{2nM} = \dfrac{\epsilon}{n}$.

So in the difference of upper sums, each case 1 occurrence contributes a 0 and each case 2 occurrence contributes at most $\epsilon/n$. But case 2 can only occur (at most) $n$-times (each occurrence must involve some new $x_k$). Thus $U(f,\mathcal{P}) - U(f,\mathcal{P}\cup\mathcal{P}') < \dfrac{\epsilon}{n}\cdot n = \epsilon$.

The proof for lower sums is analogous. ■

**Theorem:** Let $f : [a,b] \to \mathbb{R}$ be a bounded function. Then $f$ is Riemann integrable on $[a,b]$ if and only if $\underline{\int_a^b} f = \overline{\int_a^b} f$. Moreover, in such a case, $\int_a^b f = \underline{\int_a^b} f = \overline{\int_a^b} f$.

**proof:** Let $\underline{I} = \underline{\int_a^b} f = L(f)$ and $\overline{I} = \overline{\int_a^b} f = U(f)$.

Suppose $f$ be Riemann integrable on $[a,b]$ and let $I = \int_a^b f$. Suppose $\epsilon > 0$. There exists some partition $\mathcal{P}'$ such that $U(f,\mathcal{P}') - \overline{I} < \epsilon/3$ since $\overline{I}$ is a supremum. $f$ is Riemann integrable so there exists some $\delta > 0$ such that $\|\mathcal{P}\| < \delta$ implies $|\mathrm{RS}(f,\mathcal{P},\mathcal{S}) - I| < \epsilon/3$. Let $\mathcal{P}'' \supseteq \mathcal{P}'$ be a refinement such that $\|\mathcal{P}''\| < \delta$ (such refinements always exist – just union $\mathcal{P}'$ with a suitable standard partition). Thus $|\mathrm{RS}(f,\mathcal{P}'',\mathcal{S}'') - I| < \epsilon/3$. We have $\overline{I} \le U(f,\mathcal{P}'') \le U(f,\mathcal{P}')$ so that $U(f,\mathcal{P}'') - \overline{I} < \epsilon/3$. A previous lemma guarantees that there is some sampling $\mathcal{S}''$ such that $U(f,\mathcal{P}'') - \mathrm{RS}(f,\mathcal{P}'',\mathcal{S}'') < \epsilon/3$. Therefore,

$$\begin{aligned} |I - \overline{I}| &= |I - \mathrm{RS}(f,\mathcal{P}'',\mathcal{S}'') + \mathrm{RS}(f,\mathcal{P}'',\mathcal{S}'') - U(f,\mathcal{P}'') + U(f,\mathcal{P}'') - \overline{I}| \\ &\le |I - \mathrm{RS}(f,\mathcal{P}'',\mathcal{S}'')| + |\mathrm{RS}(f,\mathcal{P}'',\mathcal{S}'') - U(f,\mathcal{P}'')| + |U(f,\mathcal{P}'') - \overline{I}| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon \end{aligned}$$

This holds for all $\epsilon > 0$. Therefore, $I = \overline{I}$. Similarly, $I = \underline{I}$ so that $I = \overline{I} = \underline{I}$.

Now suppose that $\underline{I} = \overline{I}$. Call this number $I = \underline{I} = \overline{I}$. Let $\epsilon > 0$. Because the lower and upper sums are defined as a supremum and infimum, there are partitions $\mathcal{P}_1$ and $\mathcal{P}_2$ such that $U(f,\mathcal{P}_1) < I + \epsilon/2$ and $L(f,\mathcal{P}_2) > I - \epsilon/2$. Let $\mathcal{P}' = \mathcal{P}_1 \cup \mathcal{P}_2$. Then $I - \epsilon/2 < L(f,\mathcal{P}_2) \le L(f,\mathcal{P}') \le U(f,\mathcal{P}') \le U(f,\mathcal{P}_1) < I + \epsilon/2$.

Let $\delta > 0$ be the quantity given in our previous technical lemma such that for all partitions $\mathcal{P}$ with $\|\mathcal{P}\| < \delta$ we have that both $U(f,\mathcal{P}) - U(f,\mathcal{P}\cup\mathcal{P}') < \epsilon/2$ and $L(f,\mathcal{P}\cup\mathcal{P}') - L(f,\mathcal{P}) < \epsilon/2$.

So let $\mathcal{P}$ be a partition with $\|\mathcal{P}\| < \delta$ and let $\mathcal{S}$ be a sampling. Then

$$\mathrm{RS}(f,\mathcal{P},\mathcal{S}) \le U(f,\mathcal{P}) < U(f,\mathcal{P}\cup\mathcal{P}') + \frac{\epsilon}{2} \le U(f,\mathcal{P}') + \frac{\epsilon}{2} < I + \frac{\epsilon}{2} + \frac{\epsilon}{2} = I + \epsilon$$

Likewise, $I - \epsilon < \mathrm{RS}(f,\mathcal{P},\mathcal{S})$ so that $I - \epsilon < \mathrm{RS}(f,\mathcal{P},\mathcal{S}) < I + \epsilon$. We have shown that for every $\epsilon > 0$ there is some $\delta > 0$ such that for all partitions $\mathcal{P}$ with $\|\mathcal{P}\| < \delta$ and samplings $\mathcal{S}$, we have $|\mathrm{RS}(f,\mathcal{P},\mathcal{S}) - I| < \epsilon$. Therefore, $f$ is Riemann integrable with $\int_a^b f = I = \overline{I} = \underline{I}$. ■