$\{(-1)^n\}_{n=1}^{\infty}$ and $\{\sin(n)\}_{n=1}^{\infty}$ Diverge

Definition: The sequence $\{a_n\}_{n=1}^{\infty}$ *converges* to the real number L iff for every $\epsilon > 0$ there exists some $N > 0$ such that for all $n > N$ we have that $|a_n - L| < \epsilon$. We write $\lim_{n\to\infty} a_n = L$ or $a_n \to L$ and call L the *limit*[1] of the sequence. If a sequence does not converge to any real number, we say it *diverges*.

Considering the above definition, if we are to prove that a sequence diverges, we must show the L is not a limit no matter what L is. Recall that negating a logical statement flips quantifiers from $\forall$ to $\exists$ and vice-versa. Thus we must show that for every $L \in \mathbb{R}$ there is some $\epsilon > 0$ such that for every $N > 0$ there is some $n > N$ such that $|a_n - L| \geq \epsilon$.

Example: $\{(-1)^n\}_{n=1}^{\infty}$ diverges.

Proof: We will use proof by contradiction. Let $L \in \mathbb{R}$ and suppose $(-1)^n \to L$.

Before getting too much further, we will need the "ceiling" function: $\lceil N \rceil$. This is the closest integer $k = \lceil N \rceil$ such that $k \geq N$. For example, $\lceil 3.2 \rceil = \lceil \pi \rceil = 4$ and $\lceil 4.999 \rceil = \lceil 5 \rceil = 5$.

Consider $\epsilon = 1/2 > 0$. Since $(-1)^n \to L$ there exists some $N > 0$ such that $|(-1)^n - L| < \epsilon = 1/2$ for every $n > N$. Notice that both $2\lceil N \rceil$ and $2\lceil N \rceil + 1$ are greater than N. Thus $|(-1)^{2\lceil N \rceil} - L| = |1 - L| < 1/2$. This implies that $1/2 = 1 - 1/2 < L < 1 + 1/2 = 3/2$. Likewise, $|(-1)^{2\lceil N \rceil + 1} - L| = |-1 - L| < 1/2$ so that $-3/2 = -1 - 1/2 < L < -1 + 1/2 = -1/2$. This implies that L is both positive and negative (contradiction). Therefore, no such limit exists. We have that $\{(-1)^n\}_{n=1}^{\infty}$ diverges. ■

This argument can be modified to show that $\{\sin(n)\}_{n=1}^{\infty}$ diverges. The idea is that we if we go down the sequence far enough, we can hit values above $1/2$ and below $-1/2$. So the same argument (with a little finesse) will work.

Example: $\{\sin(n)\}_{n=1}^{\infty}$ diverges.

Proof: We will use proof by contradiction. Let $L \in \mathbb{R}$ and suppose $\sin(n) \to L$.

Let $\epsilon = 0.25 > 0$. There exists some $N > 0$ such that

$$|\sin(n) - L| < 0.25 \qquad \text{for all} \qquad n > N.$$

Now let's pick out values for n such that $n \geq N$ and n is as close to $\pi/2 + 2\pi k$ and $3\pi/2 + 2\pi k$ ($k \in \mathbb{Z}$) as possible (this is where sin takes on values 1 and -1 respectively). Consider

$$N_1 = \lceil \pi/2 + 2\pi\lceil N \rceil \rceil \geq \pi/2 + 2\pi N > N \qquad \text{and} \qquad N_2 = \lceil 3\pi/2 + 2\pi\lceil N \rceil \rceil \geq 3\pi/2 + 2\pi N > N$$

again where $\lceil x \rceil$ is the ceiling function. Notice that for any $x \in \mathbb{R}$, $\lceil x \rceil = x + \ell$ for some $0 \leq \ell < 1$. In particular, let $N_1 = \pi/2 + 2\pi\lceil N \rceil + \ell_1$ and $N_2 = 3\pi/2 + 2\pi\lceil N \rceil + \ell_2$ where $0 \leq \ell_1, \ell_2 < 1$.

Now $\sin(n)$ decreases on the interval $[\pi/2 + 2\pi\lceil N \rceil, 3\pi/2 + 2\pi\lceil N \rceil]$ and increases on the interval $[3\pi/2 + 2\pi\lceil N \rceil, 5\pi/2 + 2\pi\lceil N \rceil]$. Thus

$$\sin(N_1) = \sin(\pi/2 + 2\pi\lceil N \rceil + \ell_1) > \sin(\pi/2 + 2\pi\lceil N \rceil + 1) = \sin(\pi/2 + 1) \approx 0.54 > 0.5$$

and

$$\sin(N_2) = \sin(3\pi/2 + 2\pi\lceil N \rceil + \ell_2) < \sin(3\pi/2 + 2\pi\lceil N \rceil + 1) = \sin(3\pi/2 + 1) \approx -0.54 < -0.5$$

Finally, recalling $N_1, N_2 > N$ and that $|\sin(n) - L| < 0.25$ for all $n > N$, we have that

$$|\sin(N_1) - L| < 0.25 \qquad \text{and} \qquad |\sin(N_2) - L| < 0.25$$

Suppose that $L \geq 0$. This means that $|\sin(N_2) - L| = -(\sin(N_2) - L) = L - \sin(N_2) > L + 0.5 \geq 0.5$ since $\sin(N_2) < -0.5$. But this is impossible since $|\sin(N_2) - L| < 0.25$. Therefore, it cannot be the case that $L \geq 0$. Thus we must have that $L < 0$. This means that $-L > 0$ and so $|\sin(N_1) - L| = \sin(N_1) - L > 0.5 - L > 0.5$ since $\sin(N_1) > 0.5$. But this cannot be since $|\sin(N_1) - L| < 0.25$. Therefore, $L < 0$ is impossible as well. Thus L must not exist. In other words, the sequence diverges. ■

[1] **Theorem:** If a sequence converges, its limit is unique.Thus we are justified saying "the" limit not just "a" limit.