Math 3220 # Rationals and Irrationals are Densely Packed **Intro. to Real Analysis**

Theorem: Let $a, b \in \mathbb{R}$ such that $a < b$. There exists both a rational number r and an irrational number s such that $a < r < b$ and $a < s < b$. Moreover, there are infinitely many rational and irrational numbers between a and b.

Proof: First, we will prove the existence of our rational number r. To do this we consider three cases: $a \geq 0$, $a < 0 \leq b$, and finally $a < b < 0$.

- Suppose $a \geq 0$. We have that $a < b$ so that $0 < b - a$. By the squeezing in lemma, there exists some positive integer n (i.e., $n \in \mathbb{N}$) such that $0 < \frac{1}{n} < b - a$.

  Let $M = \{k \in \mathbb{N} \mid k/n > a\}$. If $a < 1/n$, then $1 \in M$. Otherwise, $0 < 1/n < a$ (*Note:* here we have used our assumption that $a \geq 0$) and the Archimedean principle states that there is some positive integer n' such that $n'/n > a$ so that $n' \in M$. Either way, we have that M is not empty. Now M is a non-empty set of positive integers. The Well-Ordering Principle states that M must have a least element. Let $m = \min(M)$ be this least element of M.

  Therefore, $\frac{m}{n} > a$ but $\frac{m-1}{n} \leq a$. Clearing denominators, we have $m - 1 \leq an$ so that $m \leq an + 1$. Therefore, $\frac{m}{n} \leq a + \frac{1}{n} < a + (b - a) = b$ (since $1/n < b - a$). Therefore, we have established that $a < r < b$ where $r = \frac{m}{n} \in \mathbb{Q}$.

- Now consider that case when $a < 0 \leq b$. We have $0 < -a$ so by the squeezing in lemma there is a positive integer n such that $0 < -\frac{1}{n} < a$. Therefore, $a < \frac{1}{n} < 0 \leq b$. Thus $r = -\frac{1}{n}$ is our desired rational number.

- Finally, consider when $a < b < 0$. Then $0 < -b < -a$. So by our first case we have some $r' \in \mathbb{Q}$ such that $-b < r' < -a$. Therefore, $a < r < b$ where $r = -r' \in \mathbb{Q}$.

Once we can squeeze in rational numbers, irrationals follow easily. First, suppose that we have two rational numbers: $\frac{u}{v}$ and $\frac{w}{x}$ (where u, v, w, and x are all integers and of course $v \neq 0$ and $x \neq 0$). Assume that $\frac{u}{v} < \frac{w}{x}$. Thus we have $\frac{ux}{vx} = \frac{u}{v} < \frac{w}{x} = \frac{vw}{vx}$ so that $ux < vw$ (because these are integers we have: $ux < ux + 1 \leq vw$).

Notice that $1 < \sqrt{2} < 2$ (since squaring gives $1 < 2 < 4$). So $0 < \sqrt{2} - 1 < 1$. Thus $ux < ux + (\sqrt{2} - 1) < ux + 1 \leq vw$. Therefore, $\frac{u}{v} = \frac{ux}{vx} < \frac{ux - 1 + \sqrt{2}}{vx} < \frac{ux + 1}{vx} \leq \frac{vw}{vx} = \frac{w}{x}$. We observe that $p = \frac{ux - 1 + \sqrt{2}}{vx}$ is irrational.

Therefore, between any two rational numbers there is an irrational number. Now consider $a < b$. We can squeeze in a rational number r_1 such that $a < r_1 < b$. Now squeeze a rational number in between r_1 and b and get $a < r_1 < r_2 < b$. Now squeeze an irrational number between those rational numbers and get $a < r_1 < p < r_2 < b$. Thus an irrational number, p, can be squeezed between any two real numbers.

We can continue such a process squeezing in rational and irrational numbers and get and endless list: $a < r_1 < p_1 < r_2 < p_2 < \cdots < b$. Thus there are an infinite number of irrational and rational numbers between any two real numbers. ■