Math 3220 # Homework #2 **Due: Mon., Feb. 4^{th}, 2019**

#1 A Selection of Tasty Sups For each of the following sets determine if its supremum and/or infimum exists. If either fails to exist, explain why. If either does exist, find it and determine if it belongs to the set or not.

Example: If $S = (-\infty, 0)$, then $\inf(S)$ does not exist since S is not bounded below. On the other hand, $\sup(S) = 0$ but $\sup(S) \notin S$ since the interval is open.

(a) $A = [0,2) \cup [4,6) \cup [8,10) \cup \cdots = \displaystyle\bigcup_{k=0}^{\infty} [4k, 4k+2)$

(b) $B = \left\{ 5 - \dfrac{2}{n} \;\middle|\; n \in \mathbb{N} \right\}$ *Note:* $\mathbb{N} = \mathbb{Z}_{>0} = \{1, 2, 3, \dots\}$

(c) $C = \left\{ -2n^2 + \dfrac{1}{(n^2+1)^2} \;\middle|\; n \in \mathbb{Z} \right\}$

Note: Sometimes doing some numerical experiments is very important when looking for a solution. Here's some Maple code which computes decimal approximations (notice the "evalf") of the members set C for integers n where $-3 \leq n \leq 5$. You could also just do this on any calculator.

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[> evalf(seq(-2*n^2+1/(n^2+1)^2,n=-3..5));
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#2 Don't Let Your Sup Get Away (Zorn 1.8 #20) Let $S \subseteq \mathbb{R}$ be non-empty and bounded above with $\beta = \sup(S)$. Show that S is *not* bounded away from β.

#3 A Supremely Infimum Problem (Zorn 1.9 #8) Let $S \subseteq \mathbb{R}$ be non-empty and bounded below. Let $-S = \{-x \mid x \in S\}$. Show that $\sup(-S)$ exists. Then show that $-\inf(S) = \sup(-S)$.

This problem shows that the completeness axiom guaranteeing the existence of supremums implies a similar statement about the existence of infimums. Write down an "infimum" version of the completeness axiom.

#4 Supreme Addition Let S and T be non-empty subsets of $\mathbb{R}$ and assume both S and T are bounded above. Let $S + T = \{s + t \mid s \in S \text{ and } t \in T\}$. Explain why $\sup(S)$, $\sup(T)$, and $\sup(S+T)$ exist. Then show that $\sup(S+T) \leq \sup(S) + \sup(T)$.

#5 Closing In (Zorn 1.9 #10) Show that every closed interval $[a, b]$ is the intersection of a nested family: $I_1 \supseteq I_2 \supseteq \cdots$ of *open* intervals.

Note: This should feel more like a calculation than a proof.

RESUBMIT Type up Homework #1 Problem #4 and its solution in LaTeX.