Math 3220 **Series Convergence Tests** **Intro. to Real Analysis**

Most all of our series will all sum from $k = 1$ to ∞. But keep in mind that "$k = 1$" could be replaced with any other integer. We typically abbreviate $\displaystyle\sum_{k=1}^{\infty} a_k$ as $\sum a_k$. We call the *sequence* $\{a_k\}_{k=1}^{\infty}$ the **terms** of the *series* $\displaystyle\sum_{k=1}^{\infty} a_k$. Also, if $s_n = a_1 + a_2 + \cdots + a_n$, then s_n is the n^{th}*partial sum* of the series $\sum a_k$. By definition, the series $\sum a_k$ converges if and only if the corresponding sequence of partial sums $\{s_n\}$ converges. Moreover, when the partial sums converge, we write $\sum a_k = \lim_{n\to\infty} s_n$. Also, keep in mind that "a finite amount of stuff doesn't effect convergence" so that. . .

Lemma: If there is some $N > 0$ such that $a_k = b_k$ for all $k > N$ (i.e., ignoring finitely many terms, both series match), then either both $\sum a_k$ and $\sum b_k$ converge or both diverge.

When just considering convergence (and not what a series sums to), the above lemma gives us permission to ignore finitely many terms in a series. We need a standard collection of series to compare other series to. Our standard examples are geometric and p-series.

Theorem: (geometric series) Let $r \in \mathbb{R}$. If $|r| < 1$,then $\displaystyle\sum_{k=0}^{\infty} r^k = \frac{1}{1-r}$. If $|r| \geq 1$, then $\sum r^k$ diverges.

Theorem: (p-series) Let $p \in \mathbb{R}$. If $p \leq 1$, then $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^p}$ diverges. If $p > 1$, then $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^p}$ converges.

Now on to the tests. . .

Theorem: (n^{th} term test) Suppose $\sum a_k$ converges. Then $a_k \to 0$. Therefore, if the sequence $\{a_k\}$ either diverges or does not converge to 0, then the series $\sum a_k$ diverges.

Theorem: (alternating series test) Suppose that $\{a_k\}$ is a monotone decreasing sequence such that $a_k \to 0$. Then $\sum (-1)^k a_k$ converges.

If $\sum |a_k|$ converges, we say that $\sum a_k$ **converges absolutely**. If $\sum a_k$ converges but $\sum |a_k|$ diverges, we say that $\sum a_k$ **converges conditionally**.

Theorem: (absolute convergence) Suppose that $\sum a_k$ converges absolutely. Then $\sum a_k$ converges. Moreover, $|\sum a_k| \leq \sum |a_k|$.

Theorem: (comparison test) Let $0 \leq a_k \leq b_k$. If $\sum b_k$ converges, then $\sum a_k$ converges. Moreover, $\sum a_k \leq \sum b_k$. On the other hand, if $\sum a_k$ diverges, then $\sum b_k$ diverges.

Theorem: (bounded comparison test) Let $\{a_k\}$ and $\{b_k\}$ be sequences of positive terms and suppose that $\{a_k/b_k\}$ is bounded above. If $\sum b_k$ converges, then $\sum a_k$ converges. On the other hand, if $\sum a_k$ diverges, then $\sum b_k$ diverges as well.

Theorem: (limit comparison test) Let $\{a_k\}$ and $\{b_k\}$ be sequences of positive terms and suppose that $a_k/b_k \to L$. If $\sum b_k$ converges, then $\sum a_k$ converges. If $L \neq 0$, $\sum a_k$ converges if and only if $\sum b_k$ converges.

Theorem: (ratio test) Let $\{a_k\}$ be a sequence of positive terms and suppose that $a_{k+1}/a_k \to L$. If $L < 1$, then $\sum a_k$ converges. If $L > 1$, then $\sum a_k$ diverges. If $L = 1$, then the test is inconclusive.

For completeness, let's mention a few other popular convergence tests. You may have seen some of these in Calculus 2.

Theorem: (integral test) Let f be a decreasing continuous non-negative function. Then $\sum_{k=1}^{\infty} f(k)$ converges if and only if the improper integral $\int_1^{\infty} f(x)\,dx$ converges.

Theorem: (root test) Let $a_k \geq 0$ and suppose that $\sqrt[k]{a_k} \to L$. If $L < 1$, then $\sum a_k$ converges. If $L > 1$, then $\sum a_k$ diverges. If $L = 1$, the test is inconclusive.

Theorem: (Dirichlet's test) Let $\{a_k\}$ be a monotone decreasing sequence such that $a_k \to 0$. Suppose that there exists some $M \in \mathbb{R}$ such that for all $N \in \mathbb{N}$ we have $\left|\sum_{k=1}^{N} b_k\right| \leq M$. Then $\sum a_k b_k$ converges.

Theorem: (Abel's test) Suppose that $\sum a_k$ converges and that $\{b_k\}$ is a monotone and bounded sequence. Then $\sum a_k b_k$ converges.

These last two tests are closely related. In practice, Abel's test is useful in many nuanced analytic arguments. Dirichlet's test subsumes the alternating series test. Notice that $\sum(-1)^k$ has bounded partial sums so if a_k is monotone decreasing to 0, Dirichlet gives us that $\sum(-1)^k a_k$ converges (i.e., the alternating series test). Dirichlet also allows us to show convergence for some very tricky oscillating series. For example, suppose that a_k is a monotone decreasing sequence which converges to zero, then we can use Dirichlet's test to conclude that $\sum \sin(k) a_k$ converges.

Notice that most of our tests are for series with either non-negative or positive terms. Because of this, most of the time we take absolute values of terms and then run a test on our new non-negative series. So generally we are really checking absolute convergence (instead of plain old convergence). Some of these tests can help when we don't have absolute convergence (for example, n-th term test, alternating series test, Dirichlet's test, and Abel's test). But generally conditionally convergent series are difficult to handle.

In some sense conditionally convergent series *almost* don't converge. While they have useful and important applications, conditionally convergent series must be handled with care. Their kind of ill behavior is why we need analysis!

Let's finish by quoting Riemann's results and thus reveal the strangeness of conditionally convergent series. First, recall that any bijection (i.e., invertible function) from $\mathbb{N} = \{1, 2, 3, \ldots\}$ to itself is called a *permuation* on $\mathbb{N}$. Let $\sigma : \mathbb{N} \to \mathbb{N}$ be a permutation on $\mathbb{N}$. Then we call $\displaystyle\sum_{k=1}^{\infty} a_{\sigma(k)}$ a **rearrangement** of $\displaystyle\sum_{k=1}^{\infty} a_k$. In other words, a rearrangement of a series is a series with the same terms. But in the rearrangement our terms are being summed in a different order. While scrambling a finite sum never changes its value (because of commutativity), scrambling infinite sums can not only change the sum's value, it can destroy convergence.

Theorem: (Riemann's rearrangment theorem) If $\sum a_k$ converges absolutely, then every rearrangement of $\sum a_k$ converges (absolutely) as well. Moreover, every rearrangement converges to the same value. On the other hand, suppose $\sum a_k$ converges conditionally. Then given any $r \in \mathbb{R}$ there is a rearrangement σ such that $\sum a_{\sigma(k)} = r$. Even more, there is a rearrangement σ such that $\sum a_{\sigma(k)} = \infty$ and one such that $\sum a_{\sigma(k)} = -\infty$. In fact, there is a rearrangement such that the partial sums $\sum a_{\sigma(k)}$ do not approach any value (finite or infinite).

This means that rearranging absolutely convergent series doesn't change how they behave. On the other hand, conditionally convergent series can be rearranged to converge to any finite or infinite value or rearranged so they straight up diverge!