Math 3510 # Homework #2 **Due: Mon., Sept. 17^{th}, 2018**

#1 Orienting Ourselves: Let C be the circle: $x^2 + z^2 = 1$ and $y = 0$.

(a) Can $\Omega = xz^2\,dx + y^3 e^z\,dy + xyz\,dz$ be an orientation for C?

(b) Give a smooth parameterization for C to show it is a 1-manifold in $\mathbb{R}^3$. Make sure you check that all of the various criterion for a smooth parameterization are met.

#2 Exterior Derivatives: Some quick computations.

(a) Let $f(x, y)$ be a smooth function and $\omega = P\,dx + Q\,dy$ both defined on $\mathbb{R}^2$. Compute df, $d\omega$, and explain why $d^2 = 0$ for forms on $\mathbb{R}^2$.

(b) Let $\omega = (x^4 + y^5 z^3)\,dy \wedge dz + xyz\,dz \wedge dx + (x^2 + y^2 + z^2)\,dx \wedge dy$. Compute $d\omega$.

#3 Into the Fourth Dimension! Consider the following "innocent" coordinate transforms...

(1)	**(2)**	**(3)**
$x = r\cos(\theta)$	$z = \rho\cos(\varphi)$	$t = \beta\cos(\alpha)$
$y = r\sin(\theta)$	$r = \rho\sin(\varphi)$	$\rho = \beta\sin(\alpha)$
$z = z$	$t = t$	$\theta = \theta$
$t = t$	$\theta = \theta$	$\varphi = \varphi$

(1) ∘ (2)	**(1) ∘ (2) ∘ (3)**
$x = \rho\sin(\varphi)\cos(\theta)$	$x = \beta\sin(\alpha)\sin(\varphi)\cos(\theta)$
$y = \rho\sin(\varphi)\sin(\theta)$	$y = \beta\sin(\alpha)\sin(\varphi)\sin(\theta)$
$z = \rho\cos(\varphi)$	$z = \beta\sin(\alpha)\cos(\varphi)$
$t = t$	$t = \beta\cos(\alpha)$

Notice that each transform is converting a pair of variables to a *polar plane*. For transform (1), we have $x^2 + y^2 = r^2$, $r \geq 0$, and $0 \leq \theta \leq 2\pi$ (or any 2π-period). For transform (2), we have $r^2 + z^2 = \rho^2$ and $\rho \geq 0$. But now $0 \leq \varphi \leq \pi$. Why only π and not 2π? Because $r \geq 0$ so $r = \rho\sin(\varphi) \geq 0$ so we need $0 \leq \varphi \leq \pi$ to keep things positive. Finally, for transform (3), $\rho^2 + t^2 = \beta^2$, $\beta \geq 0$, and $0 \leq \alpha \leq \pi$.

Putting things together, $x^2 + y^2 + z^2 + t^2 = r^2 + z^2 + t^2 = \rho^2 + t^2 = \beta^2$. Notice that (1) is a 4D cylindrical coordinate system, (2) is spherical coordinates in leaving the 4-th coordinate (i.e., t) alone, and (3) is a kind of 4D version of spherical coordinates.

The Jacobian determinant of the (1) transform is $J_1 = r$. Likewise, $J_2 = \rho$ and $J_3 = \beta$. Next, $J_{F \circ G} = J_F J_G$ (chain rule + property of determinants), so $J_{1\circ 2} = r\rho = \rho^2\sin(\varphi)$ and $J_{1\circ 2\circ 3} = r\rho\beta = \rho^2\sin(\varphi)\beta = \beta^3\sin^2(\alpha)\sin(\varphi)$.

(a) Let $B_R = \{(x, y, z, t) \in \mathbb{R}^4 \mid x^2 + y^2 + z^2 + t^2 \leq R^2\}$ (for some fixed $R > 0$).
Find the 4-volume of B_R: $\displaystyle\iiiint_{B_R} 1\,dx\,dy\,dz\,dt.$

(b) Consider the unit sphere in $\mathbb{R}^4$: $S^3 = \{(x, y, z, t) \mid x^2 + y^2 + z^2 + t^2 = 1\}$. Use the 4D spherical coordinates to parameterize S^3. Compute $\displaystyle\iiint_{S^3} t\,dx \wedge dy \wedge dz$ using your parameterization (i.e., assume that S^3 is oriented consistently with your parameterization).

#4 Verify Stokes' Theorem: Let $M = \{(x, y, z) \mid x^2 + y^2 + z^2 = 4 \text{ and } z \geq 0\}$ be the 2-manifold with orientation $\Omega = -x\,dy \wedge dz - y\,dz \wedge dx - z\,dx \wedge dy$.

(a) Identify ∂M. Then find the induced orientation on ∂M.

(b) Parameterize ∂M. Then check if your parameterization is compatible with the induced orientation found in part (a).

(c) Let $\omega = (y^2 + z^2)\,dx + x^2\,dz$. Verify the generalized Stokes' theorem for this ω and M.