Math 3510 | **Homework #3** | **Due: Wed., Sept. 26^{th}, 2018**

#1 Linear Correspondence: Find A and B given...

$$A = \begin{bmatrix} 1 & ? & 1 & ? & ? & 3 & ? & ? \\ 0 & ? & 2 & ? & ? & 1 & ? & ? \\ 2 & ? & 2 & ? & ? & -1 & ? & ? \\ -1 & ? & 1 & ? & ? & 2 & ? & ? \end{bmatrix} \overset{\text{RREF}}{\sim} \begin{bmatrix} 1 & 3 & 0 & -1 & -1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 2 & 1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} ? & ? & -2 & ? & -2 & ? \\ ? & ? & -4 & ? & 5 & ? \\ ? & ? & -6 & ? & 7 & ? \\ ? & ? & -2 & ? & 0 & ? \end{bmatrix} \overset{\text{RREF}}{\sim} \begin{bmatrix} 0 & 1 & -2 & 0 & 3 & 3 \\ 0 & 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

#2 Basis Computations: A few basis calculations.

(a) Let $\mathbf{v}_1 = (2,-3,4,-5,2)$, $\mathbf{v}_2 = (-6,9,-12,15,-6)$, $\mathbf{v}_3 = (3,-2,7,-9,1)$, $\mathbf{v}_4 = (2,-8,2,-2,6)$, $\mathbf{v}_5 = (-1,1,2,1,-3)$, $\mathbf{v}_6 = (0,-3,-18,9,12)$, $\mathbf{v}_7 = (1,0,-2,3,-2)$, and $\mathbf{v}_8 = (2,-1,1,-9,7)$.

I would suggest using Maple (or some other software to do the heavy lifting...

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with(LinearAlgebra):
A := <<2,-3,4,-5,2>|<-6,9,-12,15,-6>|<3,-2,7,-9,1>|<2,-8,2,-2,6>|
     <-1,1,2,1,-3>|<0,-3,-18,9,12>|<1,0,-2,3,-2>|<2,-1,1,-9,7>>;
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Let $W = \text{span}(S)$ where $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_8\}$. Find a subset $\beta \subseteq S$ which is a basis for W. What is $\dim(W)$? Write down β-coordinates for each vector in S.

(b) Let $W = \left\{ (x_1,x_2,x_3) \;\middle|\; \begin{array}{rcl} x_1 - 2x_2 + x_3 & = & 0 \\ 2x_1 - 3x_2 + x_3 & = & 0 \end{array} \right\}$. Invent a linear transformation $T :??? \to ???$ such that $W = \text{Ker}(T)$. Find a coordinate matrix for your T. Find a basis for W. What is $\dim(W)$?

#3 More Basis Computations: Consider $W = \left\{ \begin{bmatrix} -3a-4b+c & b-c & -4a-6b+2c \\ a+b & 2b-2c & a+b \end{bmatrix} \;\middle|\; a,b,c \in \mathbb{R} \right\}$ and

$$V = \left\{ \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \;\middle|\; \begin{array}{rcl} a+3b+2c+4d+e+7f & = & 0 \\ 2a+6b+c+5d-e+5f & = & 0 \\ 3a+9b+c+7d-2e+6f & = & 0 \end{array} \right\}.$$

(a) Both W and V are subspaces of $\mathbb{R}^{2\times 3}$. Show this is the case by re-expressing each as either a span or kernel [thus it's a subspace since spans and kernels are always subspaces]. Pick the most convenient description.

(b) Show $W \subseteq V$.

(c) Find a basis for V.

(d) Find a basis for W. Then extend this to a basis for all of V.

#4 Dimensional Issues: Let W_1 and W_2 be subspaces of a finite dimensional vector space V.

(a) Prove that $\dim(W_1 \cap W_2) \le \min\{\dim(W_1), \dim(W_2)\}$.

(b) Prove that $\dim(W_1 + W_2) \ge \max\{\dim(W_1), \dim(W_2)\}$.

Suggestion: For problems like this assume (without loss of generality = WLOG) that $\dim(W_1) = m \le \dim(W_2) = n$ so that $\min\{\dim(W_1), \dim(W_2)\} = \dim(W_1) = m$ and $\max\{\dim(W_1), \dim(W_2)\} = \dim(W_2) = n$. Always look for such *harmless* simplifying assumptions when proving a statement.

Hint: It might be helpful to remember that any linearly independent set can be extended to a basis and that every spanning set contains a basis.

#5 Quotients: Let U and W be subspaces of some vector space V. Show that $\dfrac{U+W}{W} \cong \dfrac{U}{U \cap W}$.

Hint: Apply the first isomorphism theorem to $T : U+W \to \dfrac{U}{U\cap W}$ where $T(\mathbf{u}+\mathbf{w}) = \mathbf{u} + (U \cap W)$. Things consider: Well-defined? Linear? Onto? Kernel?