#1 Dual Basis Let $\alpha = \{(1,0,0),(1,-1,0),(2,0,1)\}$.

(a) Explain why α is a basis for $\mathbb{R}^3$.

(b) Find α^* for $(\mathbb{R}^3)^*$ (i.e. find the basis dual to α).
Note: By "find" I mean give formulas for dual vectors like the formula given for f in part (c).

(c) Explain why $f \in (\mathbb{R}^3)^*$ where $f(x,y,z) = 3x+2y+z$ (what is the definition of a dual vector?). Then write f as a linear combination of α^* elements (i.e. find its α^*-coordinates).

#2 Dual Proof Let W be a subspace of a vector space V (over a field $\mathbb{F}$). We say that $f \in V^*$ **annihilates** W if $f(\mathbf{w}) = 0$ for all $\mathbf{w} \in W$. Let $A(W) = \{f \in V^* \mid f(\mathbf{w}) = 0 \text{ for all } \mathbf{w} \in W\}$ (the collection of all linear functionals which annihilate W). $A(W)$ is called the **annihilator** of W.

(a) Prove that $A(W)$ is a subspace of V^*.

(b) **[Optional:]** Suppose that $V = U \oplus W$ for some subspaces U and W. Show that $V^* = A(W) \oplus A(U)$.

(c) **[Optional:]** Let $T : V \to V$ be a linear operator and suppose that $T(W) \subseteq W$ for some subspace W (i.e. W is a T-invariant subspace). Show that $T^t(A(W)) \subseteq A(W)$ (i.e. $A(W)$ is T^t-invariant).

#3 Notational Issues Recall that according to Einstein's summation convention, the simultaneous appearance of an upper and lower index implies a summation: ${a_{imn}}^{jk}{b_{jk}}^{m\ell} = \sum\limits_m \sum\limits_j \sum\limits_k {a_{imn}}^{jk}{b_{jk}}^{m\ell} = {c_{in}}^{\ell}$.

The Levi-Civita symbol is a close companion of the Kronecker delta. The Levi-Civita symbol on 2 indices is defined by $\epsilon_{11} = \epsilon_{22} = 0$, $\epsilon_{12} = 1$, and $\epsilon_{21} = -1$. On 3 indices the symbol is defined by (for $i,j,k \in \{1,2,3\}$)...

$$\epsilon_{ijk} = \begin{cases} +1 & (i,j,k) = (1,2,3),\ (2,3,1),\ (3,1,2) \\ -1 & (i,j,k) = (1,3,2),\ (2,1,3),\ (3,2,1) \\ 0 & i = j \text{ or } j = k \text{ or } i = k \end{cases}$$

Notice that $(1,3,2)$ can be obtained from $(1,2,3)$ after a single interchange: $2^{\text{nd}} \leftrightarrow 3^{\text{rd}}$. The same is true for $(2,1,3)$ and $(3,2,1)$. In particular, $(1,3,2)$, $(2,1,3)$, and $(3,2,1)$ are all *odd* permutations of $(1,2,3)$ (we get from the identity $(1,2,3)$ to our triple after an odd number of interchanges).

On the other hand, $(2,3,1)$ requires two interchanges. From $(1,2,3)$ we interchange $1^{\text{st}} \leftrightarrow 2^{\text{nd}}$ and get $(2,1,3)$ then interchange $2^{\text{nd}} \leftrightarrow 3^{\text{rd}}$ and get $(2,3,1)$. The same is true for $(3,2,1)$. Finally, $(1,2,3)$ requires no interchanges at all. In particular, $(1,2,3)$, $(2,1,3)$, and $(3,2,1)$ are all *even* permutations of $(1,2,3)$ (we get from the identity to our triple after an even number of interchanges).

More generally, $\epsilon_{i_1 i_2 \dots i_n}$ is defined to be $+1$ if $(i_1,\dots,i_n)$ is an even permutation of $(1,\dots,n)$. It's -1 for an odd permutation and 0 if $i_k = i_\ell$ for some $k \neq \ell$ (there's a repeated index).

WARNING: We are using Einstein's summation convention in this problem.

(a) Assuming ${a_{ijk}}^{\ell m}$ and ${b_x}^{yz}$ are indexed collections of scalars where $i,j,k,\ell,m,x,y,z \in \{1,2,\dots,n\}$. Identify the appropriate sub/super-scripts on c if ${a_{ijk}}^{i\ell}{b_\ell}^{jx} = {c_{???}}^{???}$. What about ${a_{ijk}}^{\ell j}{b_\ell}^{ki}$?

(b) Let A be a 2×2 matrix with entries ${A_i}^j$. What does $\epsilon_{ij}{A_1}^i{A_2}^j$ compute? (Write this out explicitly and identify this as a familiar formula: $\epsilon_{ij}{A_1}^i{A_2}^j$ is the ??? of A.)
More generally, for a 3×3 matrix, what does $\epsilon_{ijk}{A_1}^i{A_2}^j{A_3}^k$ compute? Or for an $n \times n$ matrix, what does $\epsilon_{i_1 \dots i_n}{A_1}^{i_1} \cdots {A_n}^{i_n}$ compute? [No proof/calculation necessary. Just identify what these expression compute.]

(c) Let A be an $n \times n$ matrix with entries ${A_i}^j$. What does ${A_i}^i$ compute?