Math 3510 # Homework #5 **Due: Wed., Oct. 31^{st}, 2018**

#1 Connected and Compact For each of the following sets, decide whether they are connected and/or compact. Briefly explain your answers.

(a) $A = [-2, \infty)$

(b) $B = [-5, 2)$

(c) $C = \left\{ x \,\middle|\, 1 \leq |x| \leq 2 \right\}$

(d) $D = \left\{ (x, y) \,\middle|\, 4 \leq (x-1)^2 + (y+2)^2 \leq 9 \right\}$

(e) $E = \left\{ (x, y) \,\middle|\, |x - y| \leq 1 \right\}$

#2 Continuity Let $A \subset \mathbb{R}^n$ and $f : A \to \mathbb{R}^m$. Recall that f is continuous at $\mathbf{a} \in A$ if and only if for every $\epsilon > 0$ there is some $\delta > 0$ such that every $\mathbf{a} \in A$ (the domain) and $\|\mathbf{x} - \mathbf{a}\| < \delta$ we have $\|f(\mathbf{x}) - f(\mathbf{a})\| < \epsilon$.

Let A and B be topological spaces and $g : A \to B$. We say that g is continuous at $a \in A$ if and only if for every open set $V \subseteq B$ with $f(a) \in V$ (i.e., V is an open neighborhood of $f(a)$), there exists some open set $U \subseteq A$ such that $a \in U$ (i.e., U is an open neighborhood of a) and $f(U) \subseteq V$.

(a) Briefly explain why g is continuous (in the topology sense) if and only if g is continuous at every $a \in A$ (in the topological sense).

(b) Show that f is continuous at $\mathbf{a} \in A$ (in the analysis sense) if and only if f is continuous at $\mathbf{a} \in A$ (in the topological sense).

#3 Inverse Function Theorem The inverse function theorem says that a smooth function f with an invertible Jacobian matrix $Df(\mathbf{a})$ is a diffeomorphism when restricted to a small enough neighborhood of $\mathbf{a}$. Moreover, the Jacobian of the inverse of this restriction of f is $(Df(\mathbf{a}))^{-1}$ (the inverse of the Jacobian matrix $Df(\mathbf{a})$). In other words, when f's derivative is invertible, f is locally invertible and the derivative of the inverse is the inverse of the derivative.

Prove the following converse: Let $f : U \to V$ be a diffeomorphism (where U and V are open subsets of $\mathbb{R}^n$) and $\mathbf{a} \in U$. Show that $Df(\mathbf{a})$ is invertible.

Hint: This is really easy. Note that the chain rule says for differentiable functions $D(f \circ g) = Df \circ Dg$.

#4 Linear Stuff Let $f : \mathbb{R}^n \to \mathbb{R}^m$. Recall that f is linear if and only if $f(\mathbf{x}+\mathbf{y}) = f(\mathbf{x}) + f(\mathbf{y})$ and $f(c\mathbf{x}) = cf(\mathbf{x})$. It follows from LINEAR ALGEBRA! that $f(\mathbf{x}) = A\mathbf{x}$ for some $m \times n$ matrix A. In coordinate notation: $f^j(x^1, \ldots, x^n) = A^j_i x^i$ (implied summation).

Show that $Df = A$. This means that $Df(\mathbf{x}) = A\mathbf{x}$. So what is $D(Df)$?

Hint: This is also very easy. If you're working hard, you're overthinking this one.