

Let's explore coordinate representations of matrices and changes of basis. (Note: This problem is based on 6.4 Problem #8 – sort of.)

Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be defined by $T(x, y) = (3x + 2y, 4y)$. We will consider the relationships between matrix representations of T in coordinate systems associated with the following bases:

- $\alpha = \{(1, 0), (0, 1)\}$ (The standard basis)
- $\beta = \{(1, 1), (-2, 3)\}$
- $\beta' = \{(1, -1), (0, 1)\}$

First, let's compute $[T]_\alpha$ (the standard matrix of T), $[T]_\beta$, and $[T]_{\beta'}$.

- $T((1, 0)) = (3, 0) = 3(1, 0) + 0(0, 1)$ so that $[T((1, 0))]_\alpha = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$ and
 $T((0, 1)) = (2, 4) = 2(1, 0) + 4(0, 1)$ so that $[T((0, 1))]_\alpha = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$. Therefore,

$$[T]_\alpha = \begin{bmatrix} [T((1, 0))]_\alpha & [T((0, 1))]_\alpha \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 0 & 4 \end{bmatrix} = A$$

- $T((1, 1)) = (5, 4) = \frac{23}{5}(1, 1) - \frac{1}{5}(-2, 3)$ so that $[T((1, 1))]_\beta = \begin{bmatrix} 23/5 \\ -1/5 \end{bmatrix}$ and
 $T((-2, 3)) = (0, 12) = \frac{24}{5}(1, 1) + \frac{12}{5}(-2, 3)$ so that $[T((-2, 3))]_\beta = \begin{bmatrix} 24/5 \\ 12/5 \end{bmatrix}$. Therefore,

$$[T]_\beta = \begin{bmatrix} [T((1, 1))]_\beta & [T((-2, 3))]_\beta \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 23 & 24 \\ -1 & 12 \end{bmatrix} = B$$

- $T((1, -1)) = (1, -4) = 1(1, -1) - 3(0, 1)$ so that $[T((1, -1))]_{\beta'} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$ and
 $T((0, 1)) = (2, 4) = 2(1, -1) + 6(0, 1)$ so that $[T((0, 1))]_{\beta'} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$. Therefore,

$$[T]_{\beta'} = \begin{bmatrix} [T((1, -1))]_{\beta'} & [T((0, 1))]_{\beta'} \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -3 & 6 \end{bmatrix} = C$$

Next, let's compute change of basis matrices. To convert from β -coordinates to α -coordinates and then from β' -coordinates to α -coordinates.

- $(1, 1) = 1(1, 0) + 1(0, 1)$ so that $[(1, 1)]_\alpha = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and
 $(-2, 3) = -2(1, 0) + 3(0, 1)$ so that $[(-2, 3)]_\alpha = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$. Therefore, we can change from β to α coordinates with the following matrix:

$$[I]_\beta^\alpha = \begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix} = P$$

- $(1, -1) = 1(1, 0) - 1(0, 1)$ so that $[(1, -1)]_\alpha = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and
 $(0, 1) = 0(1, 0) + 1(0, 1)$ so that $[(0, 1)]_\alpha = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Therefore, we can change from β' to α coordinates with the following matrix:

$$[I]_{\beta'}^\alpha = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} = Q$$

- To change from β to β' coordinates we can follow: $\beta \rightarrow \alpha \rightarrow \beta'$ thus

$$[I]_\beta^{\beta'} = [I]_{\alpha}^{\beta'} [I]_\beta^\alpha = Q^{-1}P = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$$

(Or directly, $(1, 1) = 1(1, -1) + 2(0, 1)$ and $(-2, 3) = -2(1, -1) + 1(0, 1)$.)

Notice the following:

$$[T]_\beta = [I]_\alpha^\beta [T]_\alpha [I]_\beta^\alpha = P^{-1}AP = B$$

$$[T]_{\beta'} = [I]_\alpha^{\beta'} [T]_\alpha [I]_{\beta'}^\alpha = Q^{-1}AQ = C$$

$$[T]_\beta^{\beta'} = [I]_\beta^{\beta'} [T]_\beta = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \frac{1}{5} \begin{bmatrix} 23 & 24 \\ -1 & 12 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 9 & 12 \end{bmatrix}$$

or

$$[T]_\beta^{\beta'} = [I]_\alpha^{\beta'} [T]_\alpha [I]_\beta^\alpha = Q^{-1}AP = \begin{bmatrix} 5 & 0 \\ 9 & 12 \end{bmatrix}$$

Finally, let $\mathbf{v} = (-1, 3)$ then $[\mathbf{v}]_\alpha = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$ so that $[\mathbf{v}]_{\beta'} = [I]_\alpha^{\beta'} [\mathbf{v}]_\alpha = Q^{-1} \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$.

Also, $[\mathbf{v}]_\beta = [I]_\alpha^\beta [\mathbf{v}]_\alpha = P^{-1} \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix}$.

$$[T(\mathbf{v})]_{\beta'} = [T]_\beta^{\beta'} [\mathbf{v}]_\beta = \begin{bmatrix} 5 & 0 \\ 9 & 12 \end{bmatrix} \begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix} = \begin{bmatrix} 3 \\ 15 \end{bmatrix}$$

On the other hand, $T(\mathbf{v}) = T((-1, 3)) = (3, 12)$ and $[T(\mathbf{v})]_{\beta'} = [I]_\alpha^{\beta'} \begin{bmatrix} 3 \\ 12 \end{bmatrix} = Q^{-1} \begin{bmatrix} 3 \\ 12 \end{bmatrix} = \begin{bmatrix} 3 \\ 15 \end{bmatrix}$.