1 Homework. Problems pertain to material discussed between 8/22 and 8/28

Your commented presentation problems as well as a nicely written final draft of your solutions to the individual problems are due at the beginning of class on Wednesday, September 4.

Presentation Problems

You must bring a thoughtful, and complete as possible first draft of the presentation problems with you to class on ***Friday, September 30****. Everyone should have made some progress towards each problem, because at the very least you can identify pertinent definitions and/or theorems and include them in your work. However, I expect that everyone will be significantly beyond this beginning step before class on Friday. We will discuss the presentation problems during class on Friday and you should comment on your own work (please use a different color ink to make your comments). You need to turn in all of your commented presentation problems with your individual problems at the beginning of class on Wednesday, September 4th (but you should turn in exactly what you have at the end of class on Friday). I will choose one of these commented problems to grade – your grade will be based on the COMMENTS you write.*

1. (EW Exercise 1.1) Let L be a Lie algebra.
   (a) Show that $[v, 0] = 0 = [0, v]$ for all $v \in L$.
   (b) Suppose that $x, y \in L$ satisfy $[x, y] \neq 0$. Show that x and y are linearly independent over F.
2. Prove that if a Lie algebra is one-dimensional then it is Abelian.
3. Verify that the three-dimensional vector space with basis $\{x, y, z\}$ is a Lie algebra under the bilinear bracket structure $[x, y] = z$, $[x, z] = y$, $[y, z] = 0$, and $[x, x] = [y, y] = [z, z] = 0$
4. (Misra problem 1.27 ii) Consider the basis $\{E_{11}, E_{12}, E_{21}, E_{22}\}$ for the Lie algebra $gl(2, \mathbb{C})$.
   (a) Show that the subspace $span\,\{E_{11}, E_{12}\}$ is a subalgebra and describe what elements in this subalgebra look like.
   (b) Is the subspace $\{E_{11}, E_{12}, E_{21}\}$ a subalgebra? Justify your answer.

Presentation Problem Hints

1. (a) Zero is like the rice bowl empty but useful – don't forget that the bracket is bilinear.
   (b) You can use part (a) in your proof.
2. This proof really falls out from the definitions.
3. You need to check that this algebra meets the definition of a Lie algebra. Remember that you really only need to check on one ordering of the basis.
4. (a) If you want to avoid tedious matrix computations don't forget that

$$[E_{ij}, E_{k\ell}] = \delta_{jk}E_{i\ell} - \delta_{i\ell}E_{kj}$$

   (b) Same hint as in part (a)

Individual Problems

You must complete nicely written solutions to the individual problems. These solutions are due at the beginning of class on ***Wednesday, September 4.***

1. Let L be the lie algebra of 3×3 strictly upper triangular matrices under the commutator bracket

   $[A, B] = AB - BA.$

   (a) Show that L is three dimensional by giving a basis for L.

   (b) Let x be the first element in the basis that you gave in part (a), y the second and z the third. Then, the bracket structure on this Lie algebra is defined by the following relations (fill in the blank)

   $[x, y] = \_\_\_\_$
   $[x, z] = \_\_\_\_$
   $[y, z] = \_\_\_\_$

   (c) Now find $\left[\begin{pmatrix} 0 & 1 & 3 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -2 & 4 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\right]$ in two ways.

   First: directly by using matrix multiplication

   Second: By expressing each of the matrices as a linear combination of your basis vectors x, y, and z and then using the relations that you identified in part (b).

2. Let $V = F[x]$ be the polynomial ring over F. Note that V is a vector space over F. Define the linear operators on V (linear functions from V to itself) as follows

$$\begin{aligned} I(f(x)) &= f(x) \\ L_x(f(x)) &= xf(x) \\ D_x(f(x)) &= \frac{d}{dx}(f(x)) \end{aligned}$$

   Let $L = span_F \{I, L_x, D_x\}$

   (a) Show that L is a subalgebra of $gl(V)$.

   (b) If we replace L_x with $\int f(x)\ dx$ where we choose the constant of integration to be zero, would this also be a subalgebra of $gl(V)$. Explain.?

3. (Misra problem 1.27 part iv) Let $L = span \{E_{12}, E_{31}, E_{32}\} \subset gl(3, F)$.

   (a) Show that L is a subalgebra of $gl(3, F)$

   (b) Show that each $A \in L$ is nilpotent. (i.e. $A^n = 0$ for some $n \in \mathbb{Z}_{>0}$).

4. (**Only Required For Graduate Credit**–Misra 1.27 xv)

   Consider the linear transformation $T : gl(n, \mathbb{C}) \to gl(n, \mathbb{C})$ given by $T(A) = A^T$ for all $A \in gl(n, \mathbb{C})$.

   (a) Prove that 1 and -1 are the only eigenvalues of T. *Hint λ is an eigenvalue of T if and only if $T(A) = \lambda A$.*

   (b) Prove that the (-1)-eigenspace of T forms a subalgebra of $gl(n, \mathbb{C})$. What is the dimension of this subspace if $n = 2$, if $n = 3$, if $n = 4$, in general?

   (c) Is the 1-eigenspace of T a subalgebra of $gl(n, F)$? Justify.

Additional Suggested Problems (these will not be collected)

- Misra: Problem 1.27 parts (i) and (iv)
- EW: Exercises 1.2, 1.3, 1.12