Math 4010-101 # Homework #1X **Due: Fri., Nov. 8^{th}, 2013**

Let $\mathfrak{g}$ be a Lie algebra over a field $\mathbb{F}$ and let V be a $\mathfrak{g}$-module.

1. (EW page 59 Exercise 7.3) Show that V is irreducible if and only if for any $0 \neq \mathbf{v} \in V$ we have that the submodule generated by v contains all of V.

   Note: The submodule generated by $\mathbf{v}$ is the span of all elements of the form:

   $$\mathbf{x}_1 \bullet (\mathbf{x}_2 \bullet (\cdots (\mathbf{x}_\ell \bullet \mathbf{v}) \cdots)) \qquad \text{where} \quad \mathbf{x}_1, \ldots, \mathbf{x}_\ell \in \mathfrak{g} \quad \text{and} \quad \ell \geq 0.$$

   By the way, $\ell = 0$ gives us the empty product (no elements acting on $\mathbf{v}$) which is just $\mathbf{v}$ itself.

2. Basic submodule stuff.

   (a) Let W_1 and W_2 be submodules of V. Show that $W_1 \cap W_2$ and $W_1 + W_2$ are submodules. Is $W_1 \cup W_2$ a submodule? Discuss.

   (b) Let $\varphi : V \to W$ be a $\mathfrak{g}$-module homomorphism. Also, let U be a submodule of W. Show that the inverse image of U, $\varphi^{-1}(U) = \{\mathbf{v} \in V \mid \varphi(\mathbf{v}) \in U\}$, is a submodule of V which contains the kernel.

3. Let $\mathfrak{h} = \text{span}\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ be the Heisenberg Lie algebra (over $\mathbb{C}$) whose bracket structure is defined by: $[\mathbf{x}, \mathbf{y}] = \mathbf{z}$ and $\mathbf{z}$ is central ($[\mathbf{x}, \mathbf{z}] = [\mathbf{y}, \mathbf{z}] = 0$). Let $\mathbb{C}[t]$ be the algebra of polynomials with complex coefficients (with indeterminate t). For $f(t) \in \mathbb{C}[t]$ define $\mathbf{x} \bullet f(t) = f'(t)$ (differentiation), $\mathbf{y} \bullet f(t) = tf(t)$ (multiplication by t), $\mathbf{z} \bullet f(t) = f(t)$ (multiplication by 1), and extend linearly. Assuming bilinearity of this action, show $\mathbb{C}[t]$ is indeed a $\mathfrak{h}$-module. Is this an irreducible module?

4. (EW page 74 Exercise 8.3) Notice that the matrices of the form $\begin{bmatrix} \star & \star & 0 \\ \star & \star & 0 \\ 0 & 0 & 0 \end{bmatrix}$ sitting inside $\mathfrak{sl}_3(\mathbb{C})$ give us an isomorphic copy of $\mathfrak{sl}_2(\mathbb{C})$ (a similar observation allows us to treat $\mathfrak{sl}_k(\mathbb{C})$ as a subalgebra of $\mathfrak{sl}_n(\mathbb{C})$ for any $n \geq k$). This allows us to treat $V = \mathfrak{sl}_3(\mathbb{C})$ as an $\mathfrak{sl}_2(\mathbb{C})$-module with action: $\mathbf{x} \bullet \mathbf{y} = [\mathbf{x}, \mathbf{y}]$ (we are restricting the adjoint action to the subalgebra $\mathfrak{sl}_2(\mathbb{C})$). By Weyl's theorem, we know that V is completely reducible.

   Show that $V \cong V(2) \oplus V(1) \oplus V(1) \oplus V(0)$ by finding a basis of h-eigenvectors ($h = \text{diag}(1, -1, 0)$).

   [*Note:* $V(m)$ is the irreducible $\mathfrak{sl}_2(\mathbb{C})$-module of highest weight m.]

5. (EW page 75 Exercise 8.4) Suppose that V is a finite dimensional $\mathfrak{sl}_2(\mathbb{C})$-module. Using Weyl's theorem and the classification of irreducible $\mathfrak{sl}_2(\mathbb{C})$-modules, show that V is the direct sum of k irreducible submodules where $k = \dim(V_0) + \dim(V_1)$ and $V_\lambda = \{\mathbf{v} \in V \mid h \bullet \mathbf{v} = \lambda \mathbf{v}\}$.

   The rest of EW's Exercise 8.4 is false. Explain why V is **not** determined by the eigenvalues of h alone!

6. **Grad Problem** (EW page 65 Exercise 7.11) Suppose that $\varphi : \mathfrak{g} \to \mathfrak{gl}(1, \mathbb{F})$ is a 1-dimensional representation of $\mathfrak{g}$. Show that $\varphi(\mathfrak{g}') = \{\mathbf{0}\}$. [Recall that $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$.]

   Next, show that any representation of $\mathfrak{g}/\mathfrak{g}'$ can be viewed as a representation of $\mathfrak{g}$ on which $\mathfrak{g}'$ acts trivially.

   When $\mathbb{F} = \mathbb{C}$, show that if $\mathfrak{g}' \neq \mathfrak{g}$, then $\mathfrak{g}$ has infinitely many non-isomorphic 1-dimensional modules, but if $\mathfrak{g}' = \mathfrak{g}$, then the only 1-dimensional representation of $\mathfrak{g}$ is the trivial representation.