Math 4010/5530 # Homework #2 **Due: Wed., Sept. 13^{th}, 2017**

Unless otherwise specified, V is a vector space over a field $\mathbb{F}$.

#1 Coordinate Matrices Recall that when L is a Lie algebra and $x, y \in L$, we define $\text{ad}_x(y) = [x, y]$ (i.e. the adjoint map ad_x is left multiplication by x in L).

(a) Let $x \in L$. Show that $\text{ad}_x \in \mathfrak{gl}(L)$ (i.e. show that ad_x is a linear operator on L).

(b) Let $\alpha = \{E_{11}, E_{12}, E_{21}, E_{22}\}$ (in that order) be the standard basis for $\mathfrak{gl}_2(\mathbb{F})$. Find $[\text{ad}_{E_{ij}}]_\alpha$ for all i, j (i.e. compute the coordinate matrices for the adjoint operators relative to the standard basis).

(c) Let $\beta = \{E_{12}, E_{13}, E_{23}\}$ (in that order) be the standard basis for $\mathfrak{st}_3(\mathbb{F})$ (strictly upper triangular 3×3 matrices). Find $[\text{ad}_{E_{ij}}]_\beta$ for $(i, j) = (1, 2), (1, 3), (2, 3)$.

#2 Matrix Fun (Misra 1.27 part iv) Let $L = \text{span}\{E_{12}, E_{31}, E_{32}\} \subseteq \mathfrak{gl}_3(\mathbb{F})$.

(a) Show that L is a subalgebra of $\mathfrak{gl}_3(\mathbb{F})$.

(b) Let $A \in L$. Show that A is nilpotent: $A^m = 0$ for $m \gg 0$ ($m \gg 0$ means "for m sufficiently large").

#3 Differential Operators Let $V = \mathbb{F}[x]$ (i.e. polynomials with coefficients in $\mathbb{F}$). Note that V is a vector space (over $\mathbb{F}$). Define the following linear operators on V (i.e. endomorphisms on V):

$$I[f(x)] = f(x) \qquad L_x[f(x)] = xf(x) \qquad D_x[f(x)] = \frac{d}{dx}\Big[f(x)\Big]$$

(a) Let $L = \text{span}\{I, L_x, D_x\}$. Show that L is a subalgebra of $\mathfrak{gl}(V)$.

(b) If we replace L_x with $K_x[f(x)] = \displaystyle\int_0^x f(t)\,dt$ (i.e. integrate and set constant equal to zero), would this still be a subalgebra? Explain.

#4 Transpose (*for graduate students*) Let $T : \mathfrak{gl}_n(\mathbb{C}) \to \mathfrak{gl}_n(\mathbb{C})$ be defined by $T(A) = A^T$ for all $A \in \mathfrak{gl}_n(\mathbb{C})$.

(a) Show that the only eigenvalues of T are $\lambda = \pm 1$.

(b) Prove that the (-1)-eigenspace of T forms a subalgebra of $\mathfrak{gl}_n(\mathbb{C})$. What is its dimension?

(c) Is the 1-eigenspace a subalgebra? Explain.