Math 4010/5530 **Homework #6** **Due: Wed., Nov. 8th, 2017**

Let $\mathfrak{g}$ be a Lie algebra over a field $\mathbb{F}$ and let V be a $\mathfrak{g}$-module.

#1 Restriction (EW page 74 Exercise 8.3) Notice that the matrices of the form $\begin{bmatrix} \star & \star & 0 \\ \star & \star & 0 \\ 0 & 0 & 0 \end{bmatrix}$ sitting inside $\mathfrak{sl}_3(\mathbb{C})$ give us an isomorphic copy of $\mathfrak{sl}_2(\mathbb{C})$ (a similar observation allows us to treat $\mathfrak{sl}_k(\mathbb{C})$ as a subalgebra of $\mathfrak{sl}_n(\mathbb{C})$ for any $k \leq n$). This allows us to treat $V = \mathfrak{sl}_3(\mathbb{C})$ as an $\mathfrak{sl}_2(\mathbb{C})$-module with action: $\mathbf{x} \bullet \mathbf{y} = [\mathbf{x}, \mathbf{y}]$ (we are restricting the adjoint action to the subalgebra $\mathfrak{sl}_2(\mathbb{C})$). By Weyl's theorem, we know that V is completely reducible.

Show that $V \cong V(2) \oplus V(1) \oplus V(1) \oplus V(0)$ by finding a basis of h-eigenvectors ($h = \mathrm{diag}(1, -1, 0)$).

[*Note:* $V(m)$ denotes the irreducible $\mathfrak{sl}_2(\mathbb{C})$-module of highest weight m.]

#2 The Fall of Erdmann and Wildon (EW page 75 Exercise 8.4) Suppose that V is a finite dimensional $\mathfrak{sl}_2(\mathbb{C})$-module. Using Weyl's theorem and the classification of irreducible $\mathfrak{sl}_2(\mathbb{C})$-modules, show that V is the direct sum of k irreducible submodules where $k = \dim(V_0) + \dim(V_1)$ and $V_\lambda = \{\mathbf{v} \in V \mid h \bullet \mathbf{v} = \lambda \mathbf{v}\}$.

The rest of EW's Exercise 8.4 is false. Explain why V is **not** determined by the eigenvalues of h alone!

#3 (EW page 65 Exercise 7.11)

(a) Suppose that $\varphi : \mathfrak{g} \to \mathfrak{gl}(1, \mathbb{F})$ is a 1-dimensional representation of $\mathfrak{g}$. Show that $\varphi(\mathfrak{g}') = \{\mathbf{0}\}$. [Recall that $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$.]

(b) **(Grad Students)** Show that any representation of $\mathfrak{g}/\mathfrak{g}'$ can be viewed as a representation of $\mathfrak{g}$ on which $\mathfrak{g}'$ acts trivially.

(c) **(Grad Students)** When $\mathbb{F} = \mathbb{C}$, show that if $\mathfrak{g}' \neq \mathfrak{g}$, then $\mathfrak{g}$ has infinitely many non-isomorphic 1-dimensional modules, but if $\mathfrak{g}' = \mathfrak{g}$, then the only 1-dimensional representation of $\mathfrak{g}$ is the trivial representation.

Note: For 3(a) it may be worth noting that we can identify $\mathfrak{gl}(1, \mathbb{F}) = \mathbb{F}$ (these are naturally isomorphic). Having done so, in part (a) we are showing that if φ is a 1-dimensional representation (so $\varphi \in \mathfrak{g}^*$ and it is a homomorphism), then $\varphi(\mathfrak{g}') = \mathbf{0}$. It might be helpful to note that the converse is also true: if φ is a dual vector vanishing on $\mathfrak{g}'$, then φ is a 1-dimensional representation (if we identify $\mathfrak{gl}(1, \mathbb{F})$ and $\mathbb{F}$).