Math 4010-101 # Homework #7 **Due: Wed., Nov. 15^{th}, 2017**

Let $\mathfrak{g}$ be a Lie algebra over a field $\mathbb{F}$.

#1 Dualing Problem: Consider $\beta = \{(1,0,1),(1,2,0),(0,1,-1)\}$ (this is a basis for $\mathbb{R}^3$).

(a) Find the dual basis β^*. [Give formulas for each dual vector. For example: $f(x,y,z) = 2x+5y-z$. *Unnecessary note:* f is **not** one of the elements of β^*.]

(b) Explain why $f(x,y,z) = x^2$ *is not* in $(\mathbb{R}^3)^*$.

(c) Explain why $f(x,y,z) = x$ *is* in $(\mathbb{R}^3)^*$ and compute $[f]_{\beta^*}$ (this is f's β^*-coordinate vector).

(d) Find the change of basis matrices from $\text{std}^* = \{\mathbf{i}^*, \mathbf{j}^*, \mathbf{k}^*\}$ (the standard dual basis) to β^* (i.e. $[I]_{\text{std}^*}^{\beta^*}$). Also, find the change of basis matrices from to β^* to std^* (i.e. $[I]_{\beta^*}^{\text{std}^*}$).

(e) What is the relationship between $[I]_{\text{std}^*}^{\beta^*}$ and $[I]_{\text{std}}^{\beta}$?
Given two arbitrary bases α and γ for an arbitrary finite dimensional vector space (over some field $\mathbb{F}$), conjecture a relationship between $[I]_{\alpha}^{\gamma}$ and $[I]_{\alpha^*}^{\gamma^*}$.

(f) [**Grad Students**] Prove your conjecture from part (e).

#2 Look at me, Mom. I'm a module too! Let V and W be $\mathfrak{g}$-modules.

(a) The space of linear maps from V to W can be turned into a $\mathfrak{g}$-module as follows:
Let $T \in \text{Hom}(V,W) = \{S : V \to W \mid S \text{ is linear}\}$ and $g \in \mathfrak{g}$. Then for all $\mathbf{v} \in V$, define

$$(x \bullet T)(\mathbf{v}) = x \bullet T(\mathbf{v}) - T(x \bullet \mathbf{v}).$$

First, show $x \bullet T \in \text{Hom}(V,W)$. Then show that this turns $\text{Hom}(V,W)$ into a $\mathfrak{g}$-module (don't forget to check bilinearity).

(b) We can turn $\mathbb{F}$ into a $\mathfrak{g}$-module via the trivial action: $x \bullet s = 0$ for all $s \in \mathbb{F}$. Explain how this then allows us to define a dual module V^* for any $\mathfrak{g}$-module V [*Hint:* Use part (a).] What is the action of $\mathfrak{g}$ on V^*?

#3 Like Mt. Fuji Reflected in a Lake: Recall that $V(m)$ is the $\mathfrak{sl}_2(\mathbb{C})$ module with highest weight m (where m is a non-negative integer).

(a) Prove that $V(1)^* \cong V(1)$.

(b) [**Grad Students**] Prove $V(m)^* \cong V(m)$.

(c) [**Everyone**] Assuming the grad student problem (part (c)) and the fact that as $\mathfrak{g}$-modules: $\left(\bigoplus_{i=1}^{\ell} W_i\right)^* \cong \bigoplus_{i=1}^{\ell} W_i^*$ (for any finite dimensional $\mathfrak{g}$-modules W_i), prove that $V^* \cong V$ for any finite dimensional $\mathfrak{sl}_2(\mathbb{C})$-module V. **Slogan:** $\mathfrak{sl}_2$-modules are self dual!