Math 4010 # Homework #2 **Due: Mon., Sept. 8^{th}, 2025**

#1 An Ideal Question: Let R be a commutative ring with ideals I and J. Define $IJ = \{\sum_{i=1}^{n} a_i b_i \mid n \geq 0 \text{ with } a_1, \ldots, a_n \in I \text{ and } b_1, \ldots, b_n \in J\}$ where we understand that the empty sum (i.e., $n = 0$) is 0. Show IJ is an ideal.

[Grad.] Give an example of a ring R and ideals I and J such that $S = \{ab \mid a \in I \text{ and } b \in J\}$ (just products not sums of products) is *not* an ideal. *Note:* You will need to use non-principal ideals to pull this off.

#2 Unprincipaled: The end of our factorization handout claims $I = (x, 2) = \{f(x)x + g(x)2 \mid f(x), g(x) \in \mathbb{Z}[x]\}$ (this is the set of all integral polynomials with even constant term) is *not* a principal ideal of $\mathbb{Z}[x]$.

Show this directly. Is this a prime ideal? Is this a maximal ideal? Explain your answers.

#3 Prime and Maximal: As a quick reminder, in $\mathbb{Z}$ and in $\mathbb{Z}_n$, we know that subgroup = normal subgroup = cyclic subgroup = subring = ideal = principal ideal.

(a) Find all the ideals of $\mathbb{Z}_{24}$ and draw the corresponding lattice. Which ideals are prime? Which are maximal?

(b) Draw the lattice for $\mathbb{Z}_{24}/(6)$. Which are prime? Which are maximal?

#4 Quotient Issues: Let $\varphi : R \to S$ be a ring homomorphism and $I \triangleleft R$. We know $J = \varphi(I) = \{\varphi(x) \mid x \in I\} \triangleleft S$.

Prove $\overline{\varphi} : R/I \to S/J$ "defined by" $\overline{\varphi}(r + I) = \varphi(r) + J$ is a *well-defined* homomorphism.

In the case, φ is an isomorphism, show that $\overline{\varphi}$ is too.