Homework #7

- Due: Wed., Oct. 29th, 2025
- #1 The Good and the Bad: Consider the following extensions of \mathbb{Q} : $\mathbb{E}_1 = \mathbb{Q}[\sqrt{9}]$, $\mathbb{E}_2 = \mathbb{Q}[\sqrt{8}]$, and $\mathbb{E}_3 = \mathbb{Q}[\sqrt[3]{7}]$.
 - (a) Compute and carefully justify these extensions' degrees: $[\mathbb{E}_1 : \mathbb{Q}]$, $[\mathbb{E}_2 : \mathbb{Q}]$, and $[\mathbb{E}_3 : \mathbb{Q}]$.
 - (b) Explicitly write down the automorphisms (including actual formulas) making up each of the Galois groups: $G_1 = \operatorname{Gal}(\mathbb{E}_1/\mathbb{Q}), \ G_2 = \operatorname{Gal}(\mathbb{E}_2/\mathbb{Q}), \ \operatorname{and} \ G_3 = \operatorname{Gal}(\mathbb{E}_3/\mathbb{Q}).$ Carefully explain why G_1 and G_3 are trivial groups whereas G_2 is not.
 - (c) Suppose we modify the second extension so we have $\mathbb{E}_{2'} = \mathbb{F}[\sqrt{8}]$ extending \mathbb{F} for some subfield of \mathbb{C} . Is it possible that $G_{2'} = \operatorname{Gal}(\mathbb{E}_{2'}/\mathbb{F})$ has a different group structure? Can $G_{2'}$ be a bigger group? A smaller group?
- #2 Doing My Job: Let \mathbb{E}/\mathbb{F} be an extension of fields and $G = \operatorname{Gal}(\mathbb{E}/\mathbb{F})$. Let $S \subseteq G$ (a subset not necessarily a subgroup). We define $\mathbb{E}^S = \{x \in \mathbb{E} \mid \sigma(x) = x \text{ for all } \sigma \in S\}$. This is the subfield of \mathbb{E} fixed by S.
 - (a) Show \mathbb{E}^S is indeed a subfield of \mathbb{E} .
 - (b) Suppose $T \subseteq S$. Show $\mathbb{E}^T \supseteq \mathbb{E}^S$.
 - (c) [Grad.] Let H be a subgroup of G and $\sigma \in G$. Show $\sigma H \sigma^{-1}$ is a subgroup of G. Also, show $\mathbb{E}^{\sigma H \sigma^{-1}} = \sigma(\mathbb{E}^H)$. What can be said when $H \triangleleft G$?