Math 4010-101 # Homework #4 **Due: Tue., Jan. 21^{st}, 2014**

1. Suppose $a, b \in G$. We write $[a,b] = aba^{-1}b^{-1}$. This is called the *commutator* of a and b. Define:

$$G' = \langle [a,b] \mid a,b \in G \rangle = \langle aba^{-1}b^{-1} \mid a,b \in G \rangle$$

So G' is the subgroup generated by the commutators.[1] G' is called the *derived* or *commutator* subgroup of G.

(a) Prove that $G' \lhd G$ (we already know it's a subgroup, so just show it's normal). What does $G' = \{1\}$ mean?

(b) Let $N \lhd G$. Prove that G/N is Abelian if and only if $G' \subset N$.
[This means that G/G' is the largest Abelian quotient of G.]

2. Prove that S_4 does not have a subgroup isomorphic to Q (the group of Quaternions).
[*Hint:* Remember that subgroups of S_4 are either all even or half even/half odd.]

3. Let $H \leq K \leq G$ (subgroups).

(a) Assume G is finite. Prove that $[G:H] = [G:K]\,[K:H]$.

(b) [**Grad Problem**] Do part (a) again. But now without assuming G is finite.

4. Suppose that $H, K \lhd G$ and that $G = HK$ (which, since the subgroups are normal, implies that $G = KH$ as well). Prove that $G/(H \cap K) \cong G/H \times G/K$.

[1] One might ask, "Is $\{aba^{-1}b^{-1} \mid a,b \in G\}$ a subgroup?" In other words, can we avoid using "*generated by*" and just use the set of commutators? It turns out that the answer is "No." But the first example of a group whose commutators do not form a subgroup is a group of order 96.