Homework #1

Due: Fri., Jan. 22^{nd}, 2016

These problems are drawn from Rotman's "Galois Theory" (2^{nd} edition).

Page 12 #5 Show that the intersection of any family of subrings is itself a subring.

Given S_α is a subring of a ring R for all $\alpha \in I$ (I is some index set). Show $\bigcap_{\alpha \in I} S_\alpha$ is a subring of R.

Page 12 #8 Let R be a (commutative) ring (with 1) and let $f(x) = r_0 + r_1 x + \cdots + r_n x^n \in R[x]$. We can define the formal derivative[1] of $f(x)$ as follows: $f'(x) = \dfrac{d}{dx}\Big[f(x)\Big] = r_1 + 2r_2 x + \cdots + n r_n x^{n-1}$.

Prove that the derivative is linear and obeys the product rule: $[f(x)+g(x)]' = f'(x) + g'(x)$, $[cf(x)]' = cf'(x)$, and $[f(x)g(x)]' = f'(x)g(x) + f(x)g'(x)$ for all $f(x), g(x) \in R[x]$ and $c \in R$.

Keep in mind that if $f(x) = \sum_{i=0}^{n} r_i x^i$ and $g(x) = \sum_{j=0}^{m} s_j x^j$, then $f(x)g(x) = \sum_{\ell=0}^{m+n} \left(\sum_{k=0}^{\ell} r_k s_{\ell-k} \right) x^\ell$.

Page 16 #13 Degrees of Difficulty $\quad \partial(f(x)) = \deg(f(x)) =$ the degree of $f(x)$.

i. Let R be an integral domain, $f(x), g(x) \in R[x]$, and $f(x), g(x) \neq 0$. Briefly explain why the leading coefficient of $f(x)g(x)$ is the product of the leading coefficients of $f(x)$ and $g(x)$. Then justify why $\partial(f(x)g(x)) = \partial(f(x)) + \partial(g(x))$.

ii. Prove that if R is an integral domain, then so is $R[x]$.

iii. Consider $R = \mathbb{Z}_4[x]$. Show that $(2x+1)^2 = 1$. What does this say about the formula in part i and the result of part ii?

iv. Show that x can be factored: $x = f(x)g(x)$ in $\mathbb{Z}_4[x]$ in such a way that neither $f(x)$ nor $g(x)$ is constant.

Page 16 #16 Field or not a field.

i. Let $\mathbb{F}$ be a field. Show that $(\mathbb{F}[x])^\times = \mathbb{F} - \{0\}$ (i.e. the units of $\mathbb{F}[x]$ are exactly the non-zero constant polynomials).

ii. Show that $\mathbb{Z}_2[x]$ is an infinite ring with exactly 1 unit.

iii. Give an example of a non-constant polynomial in $\mathbb{Z}_4[x]$ that is a unit.

Page 17 #19 Prove that the intersection of any family of subfields is itself a subfield. (Note that this intersection isn't the trivial ring since all of the subfields contain 1.)

Let $\mathbb{E}_\alpha$ be a subfield of a field $\mathbb{F}$ for all $\alpha \in I$ (I is some index set). Show $\bigcap_{\alpha \in I} \mathbb{E}_\alpha$ is a subfield of $\mathbb{F}$.

[1] This is a totally formal notion of derivative. There is no concept of "limit" in a general ring R. Also, keep in mind that $2r_2x$ is not 2 times r_2x but instead it is the 2^{nd} additive power of r_2x. In other words, $2r_2x = r_2x + r_2x$. This may not show up in your proof, but it is something you should think about as you write up your solution.