Math 4010/5530 **Homework #2** **Due: Fri., Feb. 5th, 2016**

Problems:

Some of these problems are drawn from Rotman's "Galois Theory" (2nd edition).

Page 20 #33 Let R be a commutative ring (with 1). Show that R is a field if and only if the zero ideal, $\{0\}$ is the only *proper* ideal in R. *Note:* Don't forget to address "$1 \neq 0$" in the definition of a field.

Page 21 #34 A concrete version for undergrads and abstract for the grads.

i. Let $I = (x, 2) = \{f(x)x + g(x)2 \mid f(x), g(x \in \mathbb{Z}[x]\}$ (this is the set of all integral polynomials with even constant term). Show that I is an ideal of $\mathbb{Z}[x]$ (use the ideal test). Then show I is *not* a principal ideal.

ii. Is $\mathbb{Z}[x]$ a Euclidean domain? A PID?

Grad version: Let R be a UFD.

i. Let $a \in R$ be a non-unit and $a \neq 0$. Show that $(x, a) = \{f(x)x + g(x)a \mid f(x), g(x) \in R[x]\}$ (the ideal generated by x and a) is a non-principal ideal of $R[x]$ (x here is an indeterminate). Even through we "know" the ideal generated by x and a is an *ideal*, please prove that it is (use the ideal test). Then show it is not a principal ideal.

   Hint: Suppose that $(x, a) = (h(x))$ is principal. a is a constant polynomial belonging to $(h(x)) = (x, a)$. What can you say about the degree of $h(x)$? Next, x also belongs to $(h(x)) = (x, a)$. What does this imply? Use the fact that $R[x]$ is a UFD and that x is irreducible to get a contradiction.

ii. $R[x]$ is a PID if and only if R is a field.

Page 23 #39 Let I be an ideal of R (a ring with 1). Let S be a ring (with 1) and let $\varphi : R \to S$ be an isomorphism. Show that $J = \varphi(I) = \{\varphi(x) \mid x \in I\} \lhd S$. Then show $\bar{\varphi} : R/I \to S/J$ "defined by" $\bar{\varphi}(r + I) = \varphi(r) + J$ is a *well-defined* isomorphism.