

Irreducible Problems Let \mathbb{F} be a field.

- (a) Let $f(x) = a_n x^n + \cdots + a_1 x + a_0 \in \mathbb{F}[x]$ be irreducible. Prove that $g(x) = a_0 x^n + \cdots + a_{n-1} x + a_n$ (reverse the order of the coefficients) is irreducible in $\mathbb{F}[x]$ as well.
Note/Hint: $x^n f(1/x) = g(x)$ is called the *reciprocal polynomial* of $f(x)$.
- (b) Use part (a) and Eisenstein's criterion to show $h(x) = 12x^5 - 24x^4 + 6x^2 + 18x - 1$ is irreducible in $\mathbb{Q}[x]$.
- (c) Show $\ell(x) = 5x^3 + 9x^2 - 2x + 2$ is irreducible in $\mathbb{Q}[x]$ by reducing modulo p for some prime p .
- (d) We could also show that $\ell(x)$ is irreducible using the rational root theorem. Sketch out such a proof.
 [You don't actually have to go through the trouble of plugging numbers into $\ell(x)$.]

Page 49 #71 Find the roots of the polynomials $f(x) \in \mathbb{Z}[x]$ using the cubic / quartic formula.

- (iii) $f(x) = x^3 - 24x^2 - 24x - 25$
- (vi) **(Grad Problem)** $f(x) = x^4 - 15x^2 - 20x - 6$

Page 58 #72 Let \mathbb{E}/\mathbb{F} be an extension (of fields).

- (a) Let $\alpha, \beta \in \mathbb{E}$ be algebraic over \mathbb{F} and $\alpha \neq 0$. Prove that $\alpha + \beta$, $\alpha\beta$, and α^{-1} are also algebraic over \mathbb{F} .
[Hint: Don't try to find polynomials. Instead, use the degree formula prove $\mathbb{F}(\alpha, \beta)$ is finite dimensional over \mathbb{F} .]
- (b) Let $\mathbb{K} = \{\alpha \in \mathbb{E} \mid \alpha \text{ is algebraic over } \mathbb{F}\}$. Prove that \mathbb{K} is a subfield of \mathbb{E} containing \mathbb{F} .
- (c) **(Grad Problem)** Define $\mathbb{A} = \{\alpha \in \mathbb{C} \mid \alpha \text{ is algebraic (over } \mathbb{Q})\}$. This is called the field of *algebraic numbers*. Briefly explain why \mathbb{A}/\mathbb{Q} is an algebraic extension. Then prove it is **not** a finite extension.