Math 4010/5530

Homework #5

Due: Wed., Apr. 6th, 2016

Page 63 #79 Let $\mathbb{F}$ be a field, $f(x) \in \mathbb{F}[x]$, and $\mathbb{E}$ be a splitting field for $f(x)$ over $\mathbb{F}$. In addition, let $G = \text{Gal}(\mathbb{E}/\mathbb{F})$.

(a) Suppose that $f(x)$ is irreducible. Show that G acts **transitively** on the roots of $f(x)$ in $\mathbb{E}$. This means that given any two roots $\alpha, \beta \in \mathbb{E}$ of $f(x)$, there exists some $\sigma \in G$ such that $\sigma(\alpha) = \beta$. *Hint:* Consider the lemma and theorem on pages 55 & 56.

(b) Suppose that $f(x)$ has no repeated roots and that G acts transitively on the roots of $f(x)$ in $\mathbb{E}$. Show that $f(x)$ is irreducible. *Hint:* Suppose that $f(x)$ factors. G must send roots of a factor to other roots of that factor. Transitivity of the action will force distinct factors to share a root (why?). This is a problem (why?).

This exercise then proves that for a separable polynomial $f(x)$, G acts transitively on the roots of $f(x)$ iff $f(x)$ is irreducible.

Page 63 #80 Let $f(x) = x^4 - 10x^2 + 1 \in \mathbb{Q}[x]$. Find the Galois group of $f(x)$ (over $\mathbb{Q}$).
Note: The roots of $f(x)$ are $\pm\sqrt{2} \pm \sqrt{3}$. Rotman suggests looking at Exercise 67 (page 43) and Example 20 (page 54). My suggestive calculation: $(\sqrt{2}+\sqrt{3})^2 = 5 + 2\sqrt{6}$. $\sqrt{6}(\sqrt{2}+\sqrt{3}) = 3\sqrt{2}+2\sqrt{3}$. $3(\sqrt{2}+\sqrt{3}) - (3\sqrt{2}+2\sqrt{3}) = \sqrt{3}$.

Roots of Unity Just remember $\sqrt{1} = 1$.

(a) Find a primitive n^{th}-root of unity when $n = 1, 2, 3, 4, 5$, and 6. Your final formulas should not involve sines and cosines. For example: $e^{2\pi i/3} = \cos(2\pi/3) + i\sin(2\pi/3)$ is a primitive third root of unity, but an unacceptable (final) answer. You may find Wolfram Alpha helpful as you seek out what numbers like $\cos(2\pi/5)$ are (unless you have more special triangles memorized than I do).

(b) Find the Galois groups of $x^n - 1$ over $\mathbb{Q}$ when $n = 1, 2, 3, 4, 5, 6$ and 7.