Math 4010-101 # Homework #1 **Due: Mon., Jan. 24^{th}, 2022**

Please remember when submitting any work via email or in person to...

PUT YOUR NAME ON YOUR WORK!

#1. For each of the following determine if it is a field. If so, explain why. If not, give a **concrete** counterexample to show it fails to be a field. Also, for each of these determine its characteristic.

(a) $\mathbb{Z}[x]$ (polynomials with integer coeffs.) (b) $\mathbb{Q}[\sqrt{3}] = \{a+b\sqrt{3} \mid a, b \in \mathbb{Q}\}$ (c) $\mathbb{Z}_{10}$ (integers mod 10)

#2. The field of complex numbers $\mathbb{C}$ is *isomorphic* to the collection of matrices $S = \left\{ \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \,\middle|\, a, b \in \mathbb{R} \right\}$. Isomorphic here means that the arithmetic of $\mathbb{C}$ and S match up perfectly under the mapping $a + bi \mapsto \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$.

For example, $(a + bi) + (c + di) = (a + c) + (b + d)i$ just like $\begin{bmatrix} a & -b \\ b & a \end{bmatrix} + \begin{bmatrix} c & -d \\ d & c \end{bmatrix} = \begin{bmatrix} a + c & -(b + d) \\ b + d & a + c \end{bmatrix}$.

(a) Show that multiplication matches under this mapping.

(b) Show that multiplicative inverses match under this mapping.

(c) Recall that transposing a matrix means turnings rows into columns (and vice-versa). What does the operation of transposing a matrix in S relate to in $\mathbb{C}$?

(d) What does taking the determinant of a matrix in S relate to in $\mathbb{C}$?

Note: Technically part (c) and (d) are not features of our field isomorphism. These extra relationships show that more than just the field structure of $\mathbb{C}$ is encoded in S.

#3. Consider $W = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 \mid x_1 + x_2 + x_3 + x_4 + x_5 = 0\}$.

(a) Explain why W is a subspace of $\mathbb{R}^5$.

(b) Verify that $S = \{(2, -3, 4, -5, 2), (-6, 9, -12, 15, -6), (3, -2, 7, -9, 1), (2, -8, 2, -2, 6), (-1, 1, 2, 1, -3), (0, -3, -18, 9, 12), (1, 0, -2, 3, -2), (2, -1, 1, -9, 7)\}$ is a subset of W.

For easy copy/paste data if you compute in Maple:

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<<2,-3,4,-5,2>|<-6,9,-12,15,-6>|<3,-2,7,-9,1>|<2,-8,2,-2,6>|
<-1,1,2,1,-3>|<0,-3,-18,9,12>|<1,0,-2,3,-2>|<2,-1,1,-9,7>>;
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(c) Justify why S spans W and find a linearly independent subset of S (i.e., determine a basis for W that is contained in S).

(d) Write the remaining elements of S as linear combinations of the basis elements you found in part (c).

#4. Let $\mathbb{F}$ be some field. For each of the following sets: Explain why either this is a subspace or why it fails to be a subspace. If it is a subspace, find a basis and compute its dimension.

(a) $W_1 = \{(a_1, a_2, a_3, a_4, a_5) \in \mathbb{F}^5 \mid a_1 - a_3 - a_4 = 0\}$

(b) $W_2 = \{(a_1, a_2, a_3, a_4, a_5) \in \mathbb{F}^5 \mid a_1 = a_3 + a_5 \text{ and } a_2 - a_4 = 1\}$

(c) $W_3 = \{(a_1, a_2, a_3, a_4, a_5) \in \mathbb{F}^5 \mid a_2 = a_3 = a_4 \text{ and } a_1 + a_5 = 0\}$