**Math 4010-101** # Homework #3 **Due: Wed., Feb. $9^{\text{th}}$, 2022**

Please remember when submitting any work via email or in person to...

**PUT YOUR NAME ON YOUR WORK!**

**#1.** Coordinated Practice! Let $P_2 = \{ax^2 + bx + c \mid a, b, c \in \mathbb{R}\}$ be the subspace of $\mathbb{R}[x]$ consisting of quadratic and lower degree polynomials. Here we let $\alpha = \{1, x, x^2\}$ be our standard basis (note our intended order). Let $\mathbb{R}^{2\times 2}$ be the space of $2 \times 2$ real matrices. Let $\beta = \{E_{11}, E_{12}, E_{21}, E_{22}\}$ be our standard basis (note the ordering here – go successively across each row).

Let $T : P_2 \to \mathbb{R}^{2\times 2}$ be defined by $T(ax^2 + bx + c) = \begin{bmatrix} a+c & a+b+2c \\ 2b+2c & b+c \end{bmatrix}$.

(a) Prove $T$ is a linear transformation.

(b) Compute the standard matrix of $T$ (i.e., $[T]_\alpha^\beta$).

(c) Find a basis for both $\ker(T)$ and $T(P_2)$ – don't forget to translate back out of coordinates! What is the rank and nullity of $T$? Is $T$ one-to-one? Is it onto? Is it invertible?

(d) Let $\gamma = \{x+1, x^2+x, x^2+1\}$ and $\delta = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\}$. Justify why these are bases of $P_2$ and $\mathbb{R}^{2\times 2}$ respectively.

(e) Calculate the change of basis matrices $[I]_\alpha^\gamma$ and $[I]_\beta^\delta$.

(f) Calculate the standard coordinates of $f(x) = x^2 + 2x + 3$ and then the $\gamma$-coordinates of $f(x)$. Also, compute the standard and $\delta$-coordinates of $T(f(x))$.

(g) Find $[T]_\gamma^\delta$. Then verify that $[T(f(x))]_\delta = [T]_\gamma^\delta \cdot [f(x)]_\gamma$.

**#2.** Composing Coordinates. Recall the definitions of $P_2$, $\mathbb{R}^{2\times 2}$, $\alpha$, and $\beta$ from Problem #1 above.

Let $T = \dfrac{d}{dx} : P_2 \to P_2$ be the derivative operator on $P_2$ (i.e., $T(ax^2 + bx + c) = 2ax + b$) and let $S : \mathbb{R}^{2\times 2} \to P_2$ be defined by

$$S\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (a+b)x^2 + (a+d)x - c$$

Find $[T]_\alpha^\alpha$, $[S]_\beta^\alpha$. Then compute $[T \circ S]_\beta^\alpha$ using the previous two matrices. Verify this formula by giving a general formula for the linear transformation $T \circ S : \mathbb{R}^{2\times 2} \to P_2$ and the computing $[T \circ S]_\beta^\alpha$ directly.

**#3.** The trace map merely sums the diagonal entries of a matrix. In particular, $\text{tr} : \mathbb{F}^{n\times n} \to \mathbb{F}$ is defined by $\text{tr}(A) = \sum_{i=1}^{n} a_{ii} = a_{11} + a_{22} + \cdots + a_{nn}$ if the $(i,j)$-entry of $A$ is denoted $a_{ij}$.

(a) Prove that the trace map is linear (for general $n$).

(b) Consider the trace map on $3 \times 3$ matrices. Find its standard matrix (use a basis $\alpha$ of $E_{ij}$'s for $\mathbb{F}^{3\times 3}$ and $\beta = \{1\}$ for $\mathbb{F}$).

(c) Continuing part (b), find a basis for its kernel and range. What is its nullity and rank? Is it one-to-one, onto, and/or invertible?

**#4.** Standard Stuff. Let $T : V \to W$ be a linear transformation.

(a) Prove $\ker(T) = \{\mathbf{v} \in V \mid T(\mathbf{v}) = \mathbf{0}\} = T^{-1}(\{\mathbf{0}\})$ is a subspace of $V$ (using the subspace test – no cheating here). Then show that $T$ is one-to-one if and only if $\ker(T) = \{\mathbf{0}\}$.

(b) Suppose $V$ and $W$ are finite dimensional (e.g., $\dim(V) = n$ and $\dim(W) = m$) with bases $\alpha$ and $\beta$ respectively. Given $A = [T]_\alpha^\beta$, what about $A$ tells us whether $T$ is one-to-one or not? What would tell us whether $T$ is onto or not?

(c) Continuing part (b), explain why in the case that $\dim(V) = \dim(W)$ (for example, in the case of a linear operator) we have that $T$ is one-to-one if and only if $T$ is onto.