Math 4010-101 # Homework #4 **Due: Wed., Feb. 16^{th}, 2022**

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#1. A Based Problem Let $\text{std} = \{f_1, f_2, f_3\} = \{1, x, x^2\}$ be the standard basis for $P_2 = \{ax^2+bx+c \mid a,b,c \in \mathbb{R}\}$. Also, let $\alpha = \{g_1, g_2, g_3\} = \{1+x^2, 1+2x+x^2, 2+x\}$.

(a) Explain why α is a basis for P_2. Then find $[I]_{\text{std}}^{\alpha}$.

(b) Give explicit formulas for $f_1^*, f_2^*, f_3^*, g_1^*, g_2^*$, and g_3^*. For example, $f_3^*(ax^2+bx+c) = a$.

(c) Compute $[3x^2+2x+1]_\alpha$ using your change of basis matrix found in (a) and then using your formulas for $\alpha^* = \{g_1^*, g_2^*, g_3^*\}$ found in part (b).

(d) Let $m(ax^2+bx+c) = -4a+3c$. Find $[m]_{\text{std}^*}$ and $[m]_{\alpha^*}$.

#2. Completely Annihilated by Math Let V be a vector space over $\mathbb{F}$ and let W be a subspace of V. We define $A(W) = \{f \in V^* \mid f(w) = 0 \text{ for all } w \in W\}$. In other words, $f \in A(W)$ if $f(W) = \{0\}$ (i.e., f annihilates all of our subspace W).

(a) Show that $A(W)$ is a subspace of V^*.

[I'll do this one for you.]

First, the zero functional sends all vectors to zero (the scalar). Thus $0 \in A(W)$ (i.e., the annihilator of W is a non-empty subset of V^*). Let $f, g \in A(W)$ and $s \in \mathbb{F}$. Notice that for all $\mathbf{w} \in W$, we have $(f+g)(\mathbf{w}) = f(\mathbf{w}) + g(\mathbf{w}) = 0 + 0 = 0$ and $(sf)(\mathbf{w}) = s\,f(\mathbf{w}) = s\,0 = 0$. Thus $f+g, sf \in A(W)$. Therefore, $A(W)$ is a subspace.

(b) Suppose U is a subspace of W. Explain why $A(W) \subseteq A(U)$.

[And this one too.]

Let $f \in A(W)$. This means that $f(\mathbf{w}) = 0$ for all $\mathbf{w} \in W$. Suppose $\mathbf{u} \in U$. Then because $U \subseteq W$ we have $\mathbf{u} \in W$ and so $f(\mathbf{u}) = 0$. Therefore, f annihilates all of U and thus $f \in A(U)$. Thus $A(W) \subseteq A(U)$. Briefly, if we annihilate all of W, then since U is contained in W, we certainly annihilate all of U.

(c) Suppose $V = U \oplus W$. Show that $V^* = A(W) \oplus A(U)$.

Note: You need to show that every dual vector is a sum of a dual vector annihilating W and one annihilating U. Also, you need to show that if $f \in A(W) \cap A(U)$ then $f = 0$.

Big hint: Consider $\pi_U : V \to V$ defined by $\pi(\mathbf{u}+\mathbf{w}) = \mathbf{u}$ where $\mathbf{u} \in U$ and $\mathbf{w} \in W$ (this is well defined since every vector in V is a *unique* sum of a vector in U and a vector W – because V is a *direct sum* of those spaces). This π_U is called a projection onto U. It is linear. Likewise, define π_W. Consider composing $f \in V^*$ with these maps.

(d) Let $T : V \to V$ be a linear operator and suppose that $T(W) \subseteq W$ (i.e., W is T-invariant). Show that $T^*(A(W)) \subseteq A(W)$ (i.e., $A(W)$ is T^*-invariant).

Recall: $T^* : V^* \to V^*$ is the transpose of T defined by $T^*(f) = f \circ T$.

#3. Mapping Stuff Let $T : V \to W$ be a linear transformation.

(a) Let U be a subspace of V. Show that $T(U) = \{T(\mathbf{u}) \mid \mathbf{u} \in U\}$ is a subspace W.

Note: As a consequence, $T(V)$ (i.e., the range of T) is a subspace of W.

[I'll do this one for you.]

Notice that $\mathbf{0} \in U$ (since U is a subspace) and thus $\mathbf{0} = T(\mathbf{0}) \in T(U)$. Therefore, $T(U)$ is a non-empty subset of W. Next, let $\mathbf{x}, \mathbf{y} \in T(U)$ and $s \in \mathbb{F}$. This implies that there exists some $\mathbf{a}, \mathbf{b} \in U$ such that $T(\mathbf{a}) = \mathbf{x}$ and $T(\mathbf{b}) = \mathbf{y}$. Therefore, $\mathbf{x}+\mathbf{y} = T(\mathbf{a}) + T(\mathbf{b}) = T(\mathbf{a}+\mathbf{b}) \in T(U)$ since $\mathbf{a}+\mathbf{b} \in U$. Likewise, $s\,\mathbf{x} = s\,T(\mathbf{a}) = T(s\,\mathbf{a}) \in T(U)$ since $s\,\mathbf{a} \in U$. Thus $T(U)$ is a subspace.

(b) Let U be a subspace of W. Show that $T^{-1}(U) = \{\mathbf{v} \in V \mid T(\mathbf{v}) \in U\}$ is a subspace of V.

Note: As a consequence, $T^{-1}(\{\mathbf{0}\}) = \ker(T)$ is a subspace of V.

Hint: $\mathbf{v} \in T^{-1}(U)$ if and only if $T(\mathbf{v}) \in U$. This proof *should* be easier than the one in part (a).

(c) [The Second Isomorphism Theorem] Let U_1 and U_2 be subspaces of V. Show that $(U_1+U_2)/U_2 \cong U_1/(U_1 \cap U_2)$.

Hint: Consider the map $\mathbf{v} \mapsto \mathbf{v}+U_2$ from U_1 to $(U_1+U_2)/U_2$. Apply the First Isomorphism Theorem.