Math 4010-101 # Homework #5 **Due: Wed., Mar. 23rd, 2022**

Please remember when submitting any work via email or in person to...

PUT YOUR NAME ON YOUR WORK!

#1. Invariant, Cyclic, Minimal I'll go over some background. This may look intimidating, but the problems below are actually pretty straightforward.

Let $T : V \to V$ be a linear operator on a finite dimensional vector spaces over some field $\mathbb{F}$. Let $f(t) = a_n t^n + \cdots + a_1 t + a_0$ be a polynomial with coefficients in $\mathbb{F}$ (i.e., $f(t) \in \mathbb{F}[t]$). Recall that $f(T) : V \to V$ is a linear operator where $f(T)(\mathbf{v}) = a_n T^n(\mathbf{v}) + \cdots + a_1 T(\mathbf{v}) + a_0\mathbf{v}$ for all $\mathbf{v} \in V$ and $T^k(\mathbf{v}) = \underbrace{T(T(\cdots T(}_{k-\text{times}}\mathbf{v})\cdots))$.

There is a monic (i.e., leading coefficient is 1) polynomial, $m(t)$, such that $m(T) = 0$ (the zero operator: $(m(T))(\mathbf{v}) = \mathbf{0}$ for all $\mathbf{v} \in V$) and given any other polynomial $g(t)$ such that $g(T) = 0$, we have that $g(t)$ is a multiple of $m(t)$. This polynomial is called the *minimal polynomial* of T.

Note: One can show that the set of all polynomials $g(t)$ that annihilate T (i.e., $g(T) = 0$) form something an *ideal* in the polynomial ring $\mathbb{F}[t]$. It turns out (since $\mathbb{F}[t]$ is a principal ideal domain) that this ideal has a generator. In fact, it has a *unique* monic (= leading coefficient is 1) generator. This unique monic generator is nothing more than our minimal polynomial.

Let W be a subspace of V. We say W is T-invariant if $T(\mathbf{w}) \in W$ for all $\mathbf{w} \in W$ (i.e., $T(W) \subseteq W$).

If W is T-invariant, we can restrict T's domain to W and since $T(W) \subseteq W$, we can also cut its codomain down to W. Thus we get a map $S = T\Big|_W : W \to W$. In other words, S is a linear operator on W (instead of V) and it matches T on its domain (i.e., $S(\mathbf{w}) = T(\mathbf{w})$ for all $\mathbf{w} \in W$).

(a) Suppose W is T-invariant. Show that W is also $f(T)$-invariant.

(b) Let $\mathbf{v} \in V$ and let $W = \text{span}\{\mathbf{v}, T(\mathbf{v}), T^2(\mathbf{v}), \ldots\}$ (i.e., W is the T-cyclic subspace generated by $\mathbf{v}$). Obviously W is a subspace because it is defined as a span. Show that W is T-invariant.

(c) Again, let $W = \text{span}\{\mathbf{v}, T(\mathbf{v}), T^2(\mathbf{v}), \ldots\}$. Suppose that $T\Big|_W = S$ (the restriction of T to W) has a minimal polynomial $m(t) = t^3 - 2t + 5$. Prove that $\alpha = \{\mathbf{v}, T(\mathbf{v}), T^2(\mathbf{v})\}$ is a basis for W and find $[S]_\alpha$.
Hint: Showing α is a basis – elements of W are of the form $g(T)(\mathbf{v})$ for some polynomial $g(t)$. Consider dividing $g(t)$ by $m(t)$: there exist unique polynomials $q(t)$ and $r(t)$ such that $g(t) = q(t)m(t) + r(t)$ where either $r(t) = 0$ or the degree of $r(t)$ is less than the degree of $m(t)$.
Note: $[S]_\alpha$ is an example of something called a *companion matrix*. Companion matrices are the building blocks for rational canonical form.

#2. Eigenproblem Let $T : V \to V$ be a linear operator on a finite dimensional vector spaces over some field $\mathbb{F}$.

(a) Explain why $\lambda = 0$ is an eigenvalue if and only if T is not invertible. Then suppose T is invertible and show that if λ is an eigenvalue of T, then λ^{-1} is an eigenvalue of T^{-1}.

(b) **[Extra credit:]** Recall that $T^* : V^* \to V^*$ is the transpose map defined by $T^*(f) = f \circ T$. Show that T and T^* have the same eigenvalues.

#3. Jordan Forms If A is a 2×2 matrix (working over an algebraically closed field like the complex numbers), then A must either have one or two distinct eigenvalues. The possible Jordan forms of A look like: $J_1 = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$, $J_2 = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$, or $J_3 = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}$. These go with characteristic polynomials $f_1(t) = (t-\lambda)^2$, $f_2(t) = (t-\lambda)^2$, and $f_3 t(t) = (t-\lambda)(t-\mu)$ and minimal polynomials $m_1(t) = t - \lambda$, $m_2(t) = (t-\lambda)^2$, and $m_3(t) = (t-\lambda)(t-\mu)$. The first and third are diagonalizable while the second is not.

(a) Suppose λ, μ, and η are 3 distinct scalars. If A is 3×3 it can have eigenvalues: λ, μ, and η (all distinct); λ, μ with λ repeated twice; OR just λ repeated 3 times.
Write down all possible 3×3 Jordan forms (up to reorganizing). State each form's characteristic and minimal polynomial and whether the corresponding matrix is diagonalizable.

(b) A square matrix A is *nilpotent* if $A^k = 0$ for some positive exponent $k > 0$. Suppose A is diagonalizable, show that $A = 0$.

Hint: Eigenvalues of nilpotents are? Minimal polynomial of a diagonalizable matrix?

(c) We have a matrix B whose eigenvalues are 1 and 2. Moreover: $\dim(\ker(B-I)) = 2$, $\dim(\ker(B-I)^2) = 3$, $\dim(\ker(B-I)^3) = 3$; $\dim(\ker(B-2I)) = 2$, $\dim(\ker(B-2I)^2) = 4$, $\dim(\ker(B-2I)^3) = 4$. Write down the Jordan form of B along with B's characteristic and minimal polynomials. Is B diagonalizable? Is B invertible? What is $\det(B)$? What is $\mathrm{trace}(B)$?

#4. Computational Difficulties You probably should do these computations in Maple (or similar software). Feel free to submit a Maple file (or pdf print of such).

(a) Let $A = \begin{bmatrix} 2 & -1 & -1 & -1 & 0 & -1/2 & -1 \\ 2 & 4 & -1 & 3 & -1 & -1/2 & -1 \\ -6 & -4 & 5 & -5 & 1 & 1 & 3 \\ -4 & -3 & 3 & -3 & 2 & 2 & 3 \\ -6 & -4 & 3 & -5 & 3 & 3/2 & 3 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 8 & 5 & -5 & 7 & -2 & -5/2 & -3 \end{bmatrix}$.

Copy/paste into Maple:

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with(LinearAlgebra):
A := <<2,2,-6,-4,-6,0,8>|<-1,4,-4,-3,-4,0,5>|<-1,-1,5,3,3,0,-5>|<-1,3,-5,-3,-5,0,7>|
       <0,-1,1,2,3,0,-2>|<-1/2,-1/2,1,2,3/2,2,-5/2>|<-1,-1,3,3,3,0,-3>>;
```

Find the Jordan form of A as well as an invertible matrix P (i.e., a *transition matrix*) such that $P^{-1}AP$ is the Jordan form of A. You may use Maple to do your computations, but I want to see your work building up a basis of generalized eigenvectors (i.e. don't just use the "JordanForm" command).

(b) Let $B = \begin{bmatrix} -17 & -9 & -3 \\ 36 & 19 & 6 \\ 18 & 9 & 4 \end{bmatrix}$.

Copy/paste into Maple:

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B := <<-17,36,18>|<-9,19,9>|<-3,6,4>>;
```

As before, find the Jordan form and a corresponding transition matrix for B (show your work). Then use these matrices to find $\sqrt{B}$ (i.e., $(\sqrt{B})^2 = B$) and e^B. [**Extra credit:** Find $\cos(B)$.]