Checking on a Basis

In the course notes from Kailash C. Misra, Theorem 1.8 presents a method (without proof) for checking to see if one has a Lie algebra by examining products of basis elements. In this note, we will give a slightly adjusted version of that result with proof. First, we begin with a lemma about the Jacobi identity's symmetry and an easy consequence.

Note: We will say "the Jacobi identity holds for a, b, c" if $[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0$.

Lemma: (S_3-Symmetry of the Jacobi Identity) Given $[\cdot, \cdot] : L \times L \to L$ is a bilinear and alternating multiplication on a vector space L (over some field $\mathbb{F}$) with $a, b, c \in L$, if the Jacobi identity holds for a, b, c, then it holds for all permutations of a, b, c (i.e., it holds for a, b, c; a, c, b; b, a, c; b, c, a; c, a, b; and c, b, a).

Proof: Notice that the group of permutations on 3 characters, $S_3 = \{(1), (12), (13), (23), (123), (132)\}$, is generated by the transpositions (12) and (23): $(1) = (12)^2$, $(13) = (23)(12)(23)$, $(123) = (12)(23)$, and $(132) = (23)(12)$. Thus if we show the Jacobi identity holds for a, b, c implies it holds for both b, a, c and a, c, b, then it must hold for all other permutations as well.

Suppose it holds for a, b, c. Then, it holds for b, a, c since $[b, [a, c]] + [a, [c, b]] + [c, [b, a]] = [b, -[c, a]] + [a, -[b, c]] + [c, -[a, b]] = -[b, [c, a]] - [a, [b, c]] - [c, [a, b]] = -([a, [b, c]] + [b, [c, a]] + [c, [a, b]]) = 0$ where we used skew-symmetry to flip brackets around and linearity to pull out minus signs. Likewise, it holds for a, c, b since $[a, [c, b]] + [c, [b, a]] + [b, [a, c]] = [a, -[b, c]] + [c, -[a, b]] + [b, -[c, a]] = -[a, [b, c]] - [c, [a, b]] - [b, [c, a]] = -([a, [b, c]] + [b, [c, a]] + [c, [a, b]]) = 0$ ■

Lemma: Given $[\cdot, \cdot] : L \times L \to L$ is a bilinear and alternating multiplication on a vector space L (over some field $\mathbb{F}$) with $a, b \in L$, then the Jacobi identity holds for a, a, b; a, b, a; and b, a, a.

Proof: We need only check that it holds for a, a, b (the others follow by S_3-symmetry): $[a, [a, b]] + [a, [b, a]] + [b, [a, a]] = [a, [a, b]] + [a, -[a, b]] + [b, 0] = [a, [a, b] - [a, [a, b]] + 0 = 0$ where we used alternation to get $[a, a] = 0$ and skew symmetry to get $[b, a] = -[a, b]$ and then linearity to do the rest. ■

Now our result of interest:

Theorem: Let L be a vector space with ordered basis: $x_1, x_2, \ldots, x_n$ and suppose $[x_i, x_j]$ is defined for all $1 \leq i < j \leq n$. Moreover, this is extended assuming alternation (plus skew-symmetry) and linearly: $[x_i, x_i] = \mathbf{0}$ for all i, $[x_j, x_i] = -[x_i, x_j]$ for $i < j$, and $\left[\sum_{i=1}^n a_i x_i, \sum_{j=1}^n b_j x_j\right] = \sum_{i=1}^n \sum_{j=1}^n a_i b_j [x_i, x_j]$ for all $a_1, \ldots, a_n, b_1, \ldots, b_n \in \mathbb{F}$.

Then, L is a Lie algebra if and only if the Jacobi identity holds for x_i, x_j, x_k for all distinct ordered triples $1 \leq i < j < k \leq n$.

Proof: If L is a Lie algebra, then the Jacobi identity always holds. Conversely, suppose it holds for all distinct ordered triples. First, we will see that our bracket is alternating. Let $a_1, \ldots, a_n \in \mathbb{F}$.

$$\begin{aligned}
\left[\sum_{i=1}^n a_i x_i, \sum_{j=1}^n a_j x_j\right] &= \sum_{i=1}^n \sum_{j=1}^n a_i a_j [x_i, x_j] = \sum_{i<j} a_i a_j [x_i, x_j] + \sum_{i=j} a_i a_j [x_i, x_j] + \sum_{i>j} a_i a_j [x_i, x_j] \\
&= \sum_{i<j} a_i a_j [x_i, x_j] + \sum_{i=1}^n (a_i)^2 [x_i, x_i] + \sum_{i<j} a_j a_i [x_j, x_i] = \sum_{i<j} a_i a_j [x_i, x_j] + 0 + \sum_{i<j} a_i a_j (-[x_i, x_j]) \\
&= \sum_{i<j} a_i a_j [x_i, x_j] - \sum_{i<j} a_i a_j [x_i, x_j] = 0.
\end{aligned}$$

Note: We split the double sum over i and j into cases: $i < j$, $i = j$, and $i > j$. Next, we substituted $i = j$ in the middle and interchanging labels i and j in the last sum. After that, we used $[x_i, x_i] = 0$ (alternation on our basis) and $[x_j, x_i] = -[x_i, x_j]$ (skew-symmetry on our basis) to get that this is zero.

We now have a linear and alternating bracket and thus can apply our lemmas above. The second lemma tells us that the Jacobi identity holds for all triples with at least one repeated vector. The first lemma (S_3-symmetry) tells us that the bracket holds not just for distinct ordered triples but for all distinct triples. Putting this together, we now have that the Jacobi identity holds for all triples of basis vectors.

Our final step is to show it holds for all vectors in L. Let $a_1, \ldots, a_n, b_1, \ldots, b_n, c_1, \ldots, c_n \in \mathbb{F}$.

$$\left[\sum_{i=1}^n a_i x_i, \left[\sum_{j=1}^n b_j x_j, \sum_{k=1}^n c_k x_k\right]\right] + \left[\sum_{j=1}^n b_j x_j, \left[\sum_{k=1}^n c_k x_k, \sum_{i=1}^n a_i x_i\right]\right] + \left[\sum_{k=1}^n c_k x_k, \left[\sum_{i=1}^n a_i x_i, \sum_{j=1}^n b_j x_j\right]\right]$$

Pulling out sums (using linearity), we get:

$$= \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} a_i b_j c_k [x_i,[x_j,x_k]] + \sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{i=1}^{n} b_j c_k a_i [x_j,[x_k,x_i]] + \sum_{k=1}^{n}\sum_{i=1}^{n}\sum_{j=1}^{n} c_k a_i b_j [x_k,[x_i,x_j]]$$

Since we are summing over all i, j, k in all three terms, we can just combine the sums:

$$= \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} a_i b_j c_k ([x_i,[x_j,x_k]] + [x_j,[x_k,x_i]] + [x_k,[x_i,x_j]]) = \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} a_i b_j c_k (0) = 0$$

The penultimate equality follows since we have that the Jacobi identity holds for all triples of basis vectors. We thus have that L is equipped with a bilinear, alternating bracket which satisfies the Jacobi identity (i.e., it is a Lie algebra). ■

Example: (Cross product Lie algebra) $\mathbb{R}^3$ equipped with the cross product is a Lie algebra.

Notice that for $v = (v_1, v_2, v_3), w = (w_1, w_2, w_3) \in \mathbb{R}^3$ the cross product is defined by $v \times w = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$ $= (v_2w_3 - w_2v_3, -(v_1w_3 - w_1v_3), v_1w_2 - w_1v_2)$. Properties of the determinant guarantee that the cross product is bilinear and alternating. We only need to check the Jacobi identity. By our above theorem it suffices to check on a single triple: $\mathbf{i}, \mathbf{j}, \mathbf{k}$:

$$\mathbf{i} \times (\mathbf{j} \times \mathbf{k}) + \mathbf{j} \times (\mathbf{k} \times \mathbf{i}) + \mathbf{k} \times (\mathbf{i} \times \mathbf{j}) = \mathbf{i} \times \mathbf{i} + \mathbf{j} \times \mathbf{j} + \mathbf{k} \times \mathbf{k} = \mathbf{0} + \mathbf{0} + \mathbf{0} = \mathbf{0}$$

Example: (Very low dimensional Lie algebras) First, note that the only bilinear and alternating bracket on a zero or one-dimensional vector space must be the zero bracket. Thus there is only one Lie algebra (up to isomorphism) of dimension 1 (likewise dimension 0). If L is 2-dimensional, say with basis $\alpha = \{x, y\}$, then if we let $[x, y] = sx + ty$ for any $s, t \in \mathbb{F}$, the above theorem tells us that L is a Lie algebra as long as we demand: $[x, x] = [y, y] = 0$, $[y, x] = -[x, y] = -sx - ty$, and extend linearly. We don't have to check the Jacobi identity here since there are no *distinct* ordered triples of just two things!

Note: While all 0 and 1-dimensional Lie algebras are Abelian, 2-dimensional Lie algebras come in two flavors. Either L is Abelian (i.e., $[x, y] = 0x + 0y = 0$ so all brackets are 0) or not. If not, $sx + ty \neq 0$. Without loss of generality, say $s \neq 0$. We have that $\beta = \{w, z\}$ where $w = sx + ty$ and $z = s^{-1}y$ is still a basis (just check independence). Note that $[w, z] = [sx + ty, s^{-1}y] = ss^{-1}[x, y] + ts^{-1}[y, y] = 1(sx + ty) + ts^{-1}0 = w$. Thus every non-Abelian 2-dimensional Lie algebra has a basis $\beta = \{w, z\}$ such that $[w, z] = w$. This confirms that there are exactly two isomorphism classes of 2-dimensional Lie algebras.

As soon as we go up to 3-dimensional Lie algebras, much more can happen. In fact, we see our first simple Lie algebra (e.g., $\mathfrak{sl}_2(\mathbb{C})$)!