Math 4010/5530 **Homework #1** **Due: Fri., Jan. 27^{th}, 2023**

#1 General Algebra Let $\mathcal{A}$ be an algebra over a field $\mathbb{F}$ (i.e. $\mathcal{A}$ is a vector space over $\mathbb{F}$ equipped with a bilinear multiplication). Denote the multiplication in $\mathcal{A}$ by juxtaposition: $(a, b) \mapsto ab$. Suppose $a, b \in \mathcal{A}$. Define $L_a, R_a : \mathcal{A} \to \mathcal{A}$ by $L_a(b) = ab$ and $R_a(b) = ba$ (i.e. L_a is left multiplication by a and R_a is right multiplication by a).

(a) Show that L_a and R_a are linear operators (i.e. linear endomorphisms).

(b) Show that $\mathcal{A}$ is an associative algebra if and only if $L_x \circ R_y = R_y \circ L_x$ for all $x, y \in \mathcal{A}$.

#2 Identity Crisis Suppose that L is a Lie algebra (over some field $\mathbb{F}$). In addition, suppose that L is a unital algebra (i.e. there is some $\mathbb{1} \in L$ such that $[\mathbb{1}, x] = x = [x, \mathbb{1}]$ for all $x \in L$). Show that L Abelian (i.e. $[x, y] = \mathbf{0}$ for all $x, y \in L$). [*Hint:* Consider $[\mathbb{1}, [x, y]]$. Use the Jacobi identity.]

#3 Linearly Independent Let L be a Lie algebra (over some field $\mathbb{F}$). Let $x, y \in L$ and suppose that $[x, y] \neq \mathbf{0}$. Show that $S = \{x, y\}$ is linearly independent. What does say about 1-dimensional Lie algebras?

#4 Basis Problem Let L be a vector space (over some field $\mathbb{F}$) with basis $\beta = \{e, f, h\}$. Suppose that $[e, f] = h$, $[h, e] = 2e$, and $[h, f] = -2f$. In addition, suppose that the bracket is bilinear, alternating, and skew-symmetric (i.e. extend the definition of $[\,\cdot\,,\,\cdot\,]$ to $L \times L$ using bilinearity, alternation, and skew-symmetry).

(a) What is $[e, e]$? What is $[f, h]$?

(b) Show that this bracket makes L a Lie algebra.
[*Hint:* Use the modified Misra's Theorem 1.8 from our handout to minimize your work.]

(c) Let $E = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $F = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, and $H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. Also, let $\alpha = \{E, F, H\}$.
Prove that α is a basis for $\mathfrak{sl}_2(\mathbb{F}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \,\middle|\, a, b, c, d \in \mathbb{F} \text{ and } a + d = 0 \right\}$.

(d) Using the commutator bracket, show that $[E, F] = H$, $[H, E] = 2E$, and $[H, F] = -2F$.