Math 4010/5530 # Homework #2 **Due: Fri., Feb. 3rd, 2023**

Unless otherwise specified, V is a vector space over a field $\mathbb{F}$.

#1 Coordinate Matrices Recall that when L is a Lie algebra and $x, y \in L$, we define $\mathrm{ad}_x(y) = [x, y]$ (i.e. the adjoint map ad_x is left multiplication by x in L).

(a) Let $x \in L$. Show that $\mathrm{ad}_x \in \mathfrak{gl}(L)$ (i.e. show that ad_x is a linear operator on L).

(b) Let $\alpha = \{E_{11}, E_{12}, E_{21}, E_{22}\}$ (in that order) be the standard basis for $\mathfrak{gl}_2(\mathbb{F})$. Find $[\mathrm{ad}_{E_{ij}}]_\alpha$ for all i, j (i.e. compute the coordinate matrices for the adjoint operators relative to the standard basis).

(c) Let $\beta = \{E_{12}, E_{13}, E_{23}\}$ (in that order) be the standard basis for $\mathfrak{st}_3(\mathbb{F})$ (strictly upper triangular 3×3 matrices). Find $[\mathrm{ad}_{E_{ij}}]_\beta$ for $(i, j) = (1, 2), (1, 3), (2, 3)$.

#2 Matrix Fun (Misra 1.27 part iv) Let $L = \mathrm{span}\{E_{12}, E_{31}, E_{32}\} \subseteq \mathfrak{gl}_3(\mathbb{F})$.

(a) Show that L is a subalgebra of $\mathfrak{gl}_3(\mathbb{F})$.

(b) Let $A \in L$. Show that A is nilpotent: $A^m = 0$ for $m \gg 0$ ($m \gg 0$ means "for m sufficiently large").

#3 Differential Operators Let $V = \mathbb{F}[x]$ (i.e. polynomials with coefficients in $\mathbb{F}$). Note that V is a vector space (over $\mathbb{F}$). Define the following linear operators on V (i.e. endomorphisms on V):

$$I[f(x)] = f(x) \qquad L_x[f(x)] = xf(x) \qquad D_x[f(x)] = \frac{d}{dx}\Big[f(x)\Big]$$

(a) Let $L = \mathrm{span}\{I, L_x, D_x\}$. Show that L is a subalgebra of $\mathfrak{gl}(V)$.

Oops! Did this in class. **Proof:** We see that L is a subspace of $\mathfrak{gl}(V)$ since spans are subspaces. Thus we only need to check closure under the (commutator) bracket. As mentioned, because of bilinearity and alternation, we need only check on ordered pairs of basis elements. So it suffices to check: $[I, L_x] = I \circ L_x - L_x \circ I = L_x - L - x = 0 \in L$, $[I, D_x] = I \circ D_x - D_x \circ I = D_x - D_x = 0 \in L$, and finally $[L_x, D_x] = L_x \circ D_x - D_x \circ L_x = -I \in L$ where the last equality is justified by considering: For all $f(x) \in V = \mathbb{F}[x]$, we have $(L_x \circ D_x)(f(x)) - (D_x \circ L_x)(f(x)) = L_x(D_x(f(x))) - D_x(L_x(f(x))) = xf'(x) - D_x(xf(x)) = xf'(x) - f(x) - xf'(x) = -f(x) = -I(f(x))$.

(b) If we replace L_x with $K_x[f(x)] = \displaystyle\int_0^x f(t)\,dt$ (i.e. integrate and set constant equal to zero), would this still be a subalgebra? Explain.

Hint: Compute $M = [D_x, K_x]$. Consider what this does to 1 and x versus what a general element $N = aI + bK_x + cD_x$ ($a, b, c \in \mathbb{F}$) does. Can you make M and N match?

#4 Transpose Let $T : \mathfrak{gl}_n(\mathbb{C}) \to \mathfrak{gl}_n(\mathbb{C})$ be defined by $T(A) = A^T$ for all $A \in \mathfrak{gl}_n(\mathbb{C})$.

Note: The λ-eigenspace of a linear operator $T : V \to V$ is $E_\lambda = \ker(T - \lambda I) = \{\mathbf{x} \in V \mid T(\mathbf{x}) = \lambda\mathbf{x}\}$ (i.e., all eigenvectors with eigenvalue λ along with the zero vector). For the operator above, an $n \times n$ matrix A is in the λ-eigenspace if and only if $A^T = T(A) = \lambda A$.

(a) (*for graduate students*) Show that the only eigenvalues of T are $\lambda = \pm 1$. *Hint:* Consider T^2. What do we call the matrices in E_1? What about E_{-1}?

(b) Prove that the (-1)-eigenspace of T forms a subalgebra of $\mathfrak{gl}_n(\mathbb{C})$. [*for graduate students*: What is its dimension?]

(c) (*for graduate students*) Is the 1-eigenspace a subalgebra? Explain.