Math 4010/5530 # Homework #3 **Due: Mon., Feb. 13^{th}, 2023**

Unless otherwise specified, V is a vector space and L is a Lie algebra over some field $\mathbb{F}$.

#1 More Eigenfun Let $A, B, C, D, P \in \mathfrak{gl}_n(\mathbb{F})$. Suppose that D is a diagonal matrix with with diagonal entries $\lambda_1, \ldots, \lambda_n$ and that A is diagonalizable with eigenvalues $\lambda_1, \ldots, \lambda_n$. In addition, suppose that P is invertible.

(a) Show that E_{ij} is an eigenvector for ad_D for each $i, j = 1, \ldots, n$. What are ad_D's eigenvalues? Why is ad_D diagonalizable? *Suggestion:* First try some 2×2 examples to see how this works.

(b) Let $A = PDP^{-1}$ and $B = PCP^{-1}$. Show that $[A, B] = P[D, C]P^{-1}$.

(c) **(Grad Students)** Show that ad_A is diagonalizable.

#2 Direct Sums Recall the definition of a vector space direct sum: $L = A \oplus B$ for some subspaces A and B if $L = A + B$ (i.e. for every $\mathbf{v} \in L$ there exists some $\mathbf{a} \in A$ and $\mathbf{b} \in B$ such that $\mathbf{v} = \mathbf{a} + \mathbf{b}$) and $A \cap B = \{\mathbf{0}\}$. The circle around the plus means the sum is direct. This is equivalent to saying for every $\mathbf{v} \in L$ there is a *unique* $\mathbf{a} \in A$ and $\mathbf{b} \in B$ such that $\mathbf{v} = \mathbf{a} + \mathbf{b}$.

If in addition L is a Lie algebra, A and B are subalgebras, and $[A, B] = \{\mathbf{0}\}$, then we say $L = A \oplus B$ as Lie algebras (i.e. L is a direct sum of its subalgebras A and B). *Note:* Unfortunately, the notation "$A \oplus B$" does not reveal whether we mean a direct sum as vector spaces or as Lie algebras. We need to be told either way.

Let $\mathbb{F}$ be field of characteristic 0. Show that $\mathfrak{gl}_n(\mathbb{F}) = \mathfrak{sl}_n(\mathbb{F}) \oplus \mathbb{F}I_n$ as Lie algebras (note: $\mathbb{F}I_n$ is the set of scalar multiples of the identity matrix).

#3 An Ideal Problem Let $X \subseteq L$ (X is just a subset). Recall that $C_L(X) = \{\mathbf{v} \in L \mid [\mathbf{v}, \mathbf{x}] = \mathbf{0} \text{ for all } \mathbf{x} \in X\}$ is the centralizer of X in L.

(a) Show that $C_L(X)$ is a subalgebra of L (don't forget to show it is a subspace as well).

(b) Assume $X \triangleleft L$ (i.e., X is an ideal of the Lie algebra L). Show that $C_L(X) \triangleleft L$.

#4 Automorphism via Conjugation For some fixed $J \in \mathfrak{gl}_n(\mathbb{F})$, recall that $\mathfrak{gl}_n^J(\mathbb{F}) = \{X \in \mathfrak{gl}_n(\mathbb{F}) \mid JX + X^T J = 0\}$. Suppose that A is an <u>invertible</u> $n \times n$ matrix such that $A^T J A = J$. Show that $\varphi : \mathfrak{gl}^J(\mathbb{F}) \to \mathfrak{gl}^J(\mathbb{F})$ defined by $\varphi(X) = A^{-1}XA$ is an automorphism of $\mathfrak{gl}^J(\mathbb{F})$.

Note: The first thing you should verify is that φ actually maps from $\mathfrak{gl}^J(\mathbb{F})$ to itself.

#5 Derivations Recall $\mathrm{Der}(\mathbb{F}[x]) = \{\partial : \mathbb{F}[x] \to \mathbb{F}[x] \mid \partial \text{ is linear and } \partial(fg) = \partial(f)\,g + f\,\partial(g) \text{ for all } f, g \in \mathbb{F}[x]\}$ where $\mathbb{F}[x]$ is the algebra of polynomials with coefficients in $\mathbb{F}[x]$.

(a) Let $\partial \in \mathrm{Der}(\mathbb{F}[x])$. Show that $\partial(1) = 0$ and thus any derivation applied to a constant yields 0.

(b) Let $p(x) \in \mathbb{F}[x]$. Define $\partial(f(x)) = p(x)f'(x)$ (i.e. $\partial = p(x)\frac{d}{dx}$). Show that ∂ is a derivation on $\mathbb{F}[x]$.

(c) Compute $[p(x)\frac{d}{dx}, q(x)\frac{d}{dx}]$ where $p(x), q(x) \in \mathbb{F}[x]$.

(d) (**Grad Students**) Show that $\mathrm{Der}(\mathbb{F}[x]) = \left\{ p(x)\dfrac{d}{dx} \;\middle|\; p(x) \in \mathbb{F}[x] \right\}$
(i.e., every derivation on $\mathbb{F}[x]$ is of the form of those defined in part (b)).